John Baird,
Glasgow University.

41 Hartland Road

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SOLUTIONS OF THE EXAMPLES

IN

A TREATISE ON ALGEBRA

BY THE SAME AUTHOR.

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SOLUTIONS OF THE EXAMPLES

IN

A TREATISE ON ALGEBRA

FOURTH EDITION.

BY

CHARLES SMITH, M.A.,

MASTER OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

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PREFACE.

It is hoped that the following Solutions of the Examples in my *Treatise on Algebra* will be of use to teachers, many of whom can ill afford time to write out detailed solutions of the questions which prove too difficult for their pupils; and that it will also be of service to those students who have not the assistance of a teacher.

The solutions will, I trust, be found sufficiently full and clear. In some cases more than one solution is given.

My thanks are due to many of my friends, and particularly to Mr J. Barnard, Mr S. Wilson, Mr H. H. Holland and Rev. J. Haworth, for their kindness in reading the proof sheets.

CHARLES SMITH.

EXAMPLES 11. Pages 42—45.

11.

$$\begin{array}{l} x^3 - 5x^2 + 1 \\ 2x^3 + 5x + 1 \\ \hline 2x^6 - 10x^5 \quad + 2x^3 \\ \qquad 5x^4 - 25x^3 \quad + 5x \\ \qquad\qquad x^3 - 5x^2 \quad + 1 \\ \hline 2x^6 - 10x^5 + 5x^4 - 22x^3 - 5x^2 + 5x + 1 \end{array}$$

12.

$$\begin{array}{l} 2x^3 - 5x^2y + y^3 \\ 2x^3 + 5xy^2 + y^3 \\ \hline 4x^6 - 10x^5y \quad + 2x^3y^3 \\ \qquad 10x^4y^2 - 25x^3y^3 \quad + 5xy^5 \\ \qquad\qquad + 2x^3y^3 - 5x^2y^4 \quad + y^6 \\ \hline 4x^6 - 10x^5y + 10x^4y^2 - 21x^3y^3 - 5x^2y^4 + 5xy^5 + y^6 \end{array}$$

13.

$$\begin{array}{l} 3a^3 - 2a^2b + 3ab^2 - 3b^3 \\ 2a^3 + 5a^2b - 4ab^2 + b^3 \\ \hline 6a^6 - 4a^5b + 6a^4b^2 - 6a^3b^3 \\ \qquad 15a^5b - 10a^4b^2 + 15a^3b^3 - 15a^2b^4 \\ \qquad\quad - 12a^4b^2 + 8a^3b^3 - 12a^2b^4 + 12ab^5 \\ \qquad\qquad\quad 3a^3b^3 - 2a^2b^4 + 3ab^5 - 3b^6 \\ \hline 6a^6 + 11a^5b - 16a^4b^2 + 20a^3b^3 - 29a^2b^4 + 15ab^5 - 3b^6 \end{array}$$

14.

$$\begin{array}{l} 2a^3x^3 - 3a^2x^2y^2 + 5y^6 \\ a^3x^3 + 4axy^4 - 2y^6 \\ \hline 2a^6x^6 - 3a^5x^5y^2 \quad + 5a^3x^3y^6 \\ \qquad 8a^4x^4y^4 - 12a^3x^3y^6 \quad + 20axy^{10} \\ \qquad\qquad - 4a^3x^3y^6 + 6a^2x^2y^8 \quad - 10y^{12} \\ \hline 2a^6x^6 - 3a^5x^5y^2 + 8a^4x^4y^4 - 11a^3x^3y^6 + 6a^2x^2y^8 + 20axy^{10} - 10y^{12} \end{array}$$

15.

$$\begin{array}{l} 2a - 3a^2 + 5a^3 - 7a^5 \\ 1 - 2a^2 + 6a^4 \\ \hline 2a - 3a^2 + 5a^3 \quad - 7a^5 \\ \qquad - 4a^3 + 6a^4 - 10a^5 \quad + 14a^7 \\ \qquad\qquad + 12a^5 - 18a^6 + 30a^7 - 42a^9 \\ \hline 2a - 3a^2 + a^3 + 6a^4 - 5a^5 - 18a^6 + 44a^7 - 42a^9 \end{array}$$

16. $a^2 - ab - ac + b^2 - bc + c^2$
$a + b + c$

$a^3 - a^2b - a^2c + ab^2 - abc + ac^2$
$+ a^2b - ab^2 - abc + b^3 - b^2c + bc^2$
$+ a^2c - abc - ac^2 + b^2c - bc^2 + c^3$

$a^3 - 3abc + b^3 + c^3$

17. $x^2 - xy - xz + y^2 - yz + z^2$
$x + y + z$

$x^3 - x^2y - x^2z + xy^2 - xyz + xz^2$
$+ x^2y - xy^2 - xyz + y^3 - y^2z + yz^2$
$+ x^2z - xyz - xz^2 + y^2z - yz^2 + z^3$

$x^3 - 3xyz + y^3 + z^3$

18. $4a^2 - 6ab + 2ac + 9b^2 + 3bc + c^2$
$2a + 3b - c$

$8a^3 - 12a^2b + 4a^2c + 18ab^2 + 6abc + 2ac^2$
$+ 12a^2b - 18ab^2 + 6abc + 27b^3 + 9b^2c + 3bc^2$
$- 4a^2c + 6abc - 2ac^2 - 9b^2c - 3bc^2 - c^3$

$8a^3 + 18abc + 27b^3 - c^3$

19. $(x^4+1)(x^2+1)(x^2-1) = (x^4+1)(x^4-1) = x^8 - 1.$

20. $(x^4+16y^4)(x^2+4y^2)(x+2y)(x-2y) = (x^4+16y^4)(x^2+4y^2)(x^2-4y^2)$
$= (x^4+16y^4)(x^4-16y^4) = x^8 - 256y^8.$

21. Product $= \{(x-y)(x+y)(x^2+y^2)\}^2 = (x^4-y^4)^2 = x^8 - 2x^4y^4 + y^8.$

22. Product $= \{(x^2+1)(x+1)(x-1)\}^3 = (x^4-1)^3 = x^{12} - 3x^8 + 3x^4 - 1.$

23. $(x^2 - x + 1)(x^2 + x + 1) = x^4 + x^2 + 1;$
$\therefore (x^4 + x^2 + 1)(x^4 - x^2 + 1) = x^8 + x^4 + 1.$

24. $(a^2 + 2ab + 4b^2)(a^2 - 2ab + 4b^2) = a^4 + 4a^2b^2 + 16b^4,$
and $(a^4 + 4a^2b^2 + 16b^4)(a^4 - 4a^2b^2 + 16b^4) = a^8 + 16a^4b^4 + 256b^8.$

25. (i) $a^2 + 4b^2 + 9c^2 + 2a\,(2b) + 2a\,(-3c) + 2\,(2b)\,(-3c)$
$= a^2 + 4b^2 + 9c^2 + 4ab - 6ac - 12bc.$

(ii) $a^4 + a^2b^2 + b^4 + 2a^2\,(-ab) + 2a^2b^2 + 2\,(-ab)\,b^2$
$= a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$

(iii) $b^2c^2 + c^2a^2 + a^2b^2 + 2bc \,.\, ca + 2bc \,.\, ab + 2ca \,.\, ab$
$= b^2c^2 + c^2a^2 + a^2b^2 + 2abc\,(a + b + c).$

(iv) $1 + 4x^2 + 9x^4 + 2\,(-2x) + 2\,(3x^2) + 2\,(-2x)\,3x^2$
$= 1 - 4x + 10x^2 - 12x^3 + 9x^4.$

(v) $x^6 + x^4 + x^2 + 1 + 2x^3 \,.\, x^2 + 2x^3 \,.\, x + 2x^3 + 2x^2 \,.\, x + 2x^2 + 2x$
$= x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1.$

26. As in Art. 67, we have $(a+b+c)^3 = \Sigma a^3 + 3\Sigma a^2 b + 6abc$.

Hence $(2a-3b-2c)^3$

$= (2a)^3 + (-3b)^3 + (-2c)^3 + 3(2a)^2(-3b) + 3(2a)^2(-2c) + 3(2a)(-3b)^2$
$+ 3(2a)(-2c)^2 + 3(-3b)^2(-2c) + 3(-3b)(-2c)^2 + 6(2a)(-3b)(-2c)$
$= 8a^3 - 27b^3 - 8c^3 - 36a^2b - 24a^2c + 54ab^2 + 24ac^2 - 54b^2c - 36bc^2 + 72abc.$

Also $(1+x+x^2)^3$

$= 1 + x^3 + x^6 + 3x + 3x^2 + 3x^2 + 3x^4 + 3x^4 + 3x^5 + 6x^3$
$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6.$

27. We have

$x^2+y^2+z^2+2yz+2zx+2xy - (x^2+y^2+z^2+2yz-2zx-2xy)$
$+ (x^2+y^2+z^2-2yz+2zx-2xy) - (x^2+y^2+z^2-2yz-2zx+2xy) = 8zx.$

28. $(x+y)(x+z) - x^2 = yz + zx + xy = \&c.$

29. Left side

$= y^2 + 2yz + z^2 + z^2 + 2zx + x^2 + x^2 + 2xy + y^2 - x^2 - y^2 - z^2$
$= x^2 + y^2 + z^2 + 2yz + 2zx + 2xy = (x+y+z)^2.$

31. On expansion it will be found that both expressions

$= 3(yz^2 - y^2z + zx^2 - z^2x + xy^2 - x^2y).$

32. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$;

$\therefore (a+b)^3 - 3ab(a+b) = a^3 + b^3.$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = a^4 + b^4 + 4ab(a^2 + b^2 + 2ab) - 2a^2b^2$;

$\therefore (a+b)^4 - 4ab(a+b)^2 + 2a^2b^2 = a^4 + b^4.$

33. $(x^2+xy+y^2)^2 - 4xy(x^2+y^2) = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$
$- 4x^3y - 4xy^3 = (x^2 - xy + y^2)^2.$

34. The expression on the left is easily seen to be

$4x^2 + 4y^2 + 4z^2 + 8yz + 8zx + 8xy = 4(x+y+z)^2.$

36. $2(x^2+y^2+z^2-yz-zx-xy) = (y-z)^2 + (z-x)^2 + (x-y)^2$
$= (b-c)^2 + (c-a)^2 + (a-b)^2 = 2(a^2+b^2+c^2-bc-ca-ab).$

37. $2(x^2+y^2+z^2-yz-zx-xy) = (y-z)^2 + (z-x)^2 + (x-y)^2$
$= (c-b)^2 + \ldots = 2(a^2+b^2+c^2-bc-ca-ab).$

38. $(b-c) + (c-a) + (a-b) = 0;$

$\therefore (b-c)^2 + (c-a)^2 + (a-b)^2 + 2(b-c)(c-a) + 2(b-c)(a-b)$
$+ 2(c-a)(a-b) = 0;$

$\therefore (b-c)^2 + (c-a)^2 + (a-b)^2 = 2(a-b)(a-c) + 2(b-c)(b-a) + 2(c-a)(c-b).$

39. Left side

$= x^2(b^2+c^2) + y^2(c^2+a^2) + z^2(a^2+b^2) - 2bcyz - 2cazx - 2abxy$

$= (b^2z^2 - 2bcyz + c^2y^2) + (c^2x^2 - 2cazx + a^2z^2) + (a^2y^2 - 2abxy + b^2x^2)$

$= (bz-cy)^2 + (cx-az)^2 + (ay-bx)^2.$

40. $ax+by+cz = a(a^2-bc) + b(b^2-ca) + c(c^2-ab) = a^3+b^3+c^3-3abc$

$= (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$ [See 16] $= (a+b+c)(x+y+z).$

Again

$bc(x^2-yz) = bc\{(a^2-bc)^2 - (b^2-ca)(c^2-ab)\} = abc(a^3+b^3+c^3-3abc),$

and the symmetry of this last result shews that it is also equal to $ca(y^2-zx)$ and to $ab(z^2-xy)$.

41. $(x-a)^3 + (x-b)^3 + (x-c)^3 - 3(x-a)(x-b)(x-c)$

$= (x-a+x-b+x-c)\{(x-a)^2 + (x-b)^2 + (x-c)^2 - (x-b)(x-c) - (x-c)(x-a) - (x-a)(x-b)\}$ [From 16, where $x-a$, $x-b$, $x-c$ are substituted for a, b, c respectively.]

$= (3x-a-b-c)\{(x-a)^2 + \&c.\} = 0,$ since $3x-a-b-c = 0.$

43. On the right, the coefficient of x^2 is a^2+ab+b^2, the coefficient of xy is $-2ab + a(a+b) - b^2 + 2b(a+b) = a^2+ab+b^2$, and the coefficient of y^2 is $b^2 - b(a+b) + (a+b)^2 = a^2+ab+b^2$.

Hence the whole expression on the right $= (a^2+ab+b^2)(x^2+xy+y^2).$

50. The product can only contain terms of which a^3, a^2b and abc are types.

The coefficients of all terms of the type a^3 are obviously unity.

The term a^2b arises in two ways, namely $a^2 \times b$ and $-ab \times a$: the coefficient of a^2b, and of every term of the same form, is therefore zero.

The term abc arises in three ways, namely $-bc \times a$, $-ca \times b$ and $-ab \times c$: the coefficient of abc, and of every term of the same form, is therefore -3.

The required product is therefore $\Sigma a^3 - 3\Sigma abc$.

III. Pages 51—52.

6.

$$
\begin{array}{l|l}
x^5 - x^4y & x^4 + x^3y + x^2y^2 + xy^3 - 4y^4 \\
\hline
x^4y & \\
x^4y - x^3y^2 & \\
\hline
x^3y^2 & \\
x^3y^2 - x^2y^3 & \\
\hline
x^2y^3 & \\
x^2y^3 - xy^4 & \\
\hline
-4xy^4 & \\
-4xy^4 + 4y^5 & \\
\hline
\end{array}
\qquad -5xy^4 + 4y^5
$$

7. $$1-2+1\,)\,1+0+0+0+0-6+5\,(\,1+2+3+4+5$$
$$1-2+1$$
$$2-1$$
$$2-4+2$$
$$3-2$$
$$3-6+3$$
$$4-3$$
$$4-8+4$$
$$5-10+5$$
$$5-10+5$$

Ans. $1+2x+3x^2+4x^3+5x^4$.

8. As in 7. *Ans.* $m^4+2m^3n+3m^2n^2+4mn^3+5n^4$.

9. $$1-2+1\,)\,1+0+0+0+0+0-7+6\,(\,1+2+3+4+5+6$$
$$1-2+1$$
$$2-1$$
$$2-4+2$$
$$3-2$$
$$3-6+3$$
$$4-3$$
$$4-8+4$$
$$5-4$$
$$5-10+5$$
$$6-12+6$$
$$6-12+6$$

Ans. $1+2x+3x^2+4x^3+5x^4+6x^5$.

14. $$x+y-1\,)\,x^3-3x^2+3x+y^3-1\,(\,x^2-x(y+2)+y^2+y+1$$
$$x^3+x^2(y-1)$$
$$-x^2(y+2)$$
$$-x^2(y+2)-x(y+2)(y-1)$$
$$x(y^2+y+1)+y^3-1$$
$$x(y^2+y+1)+(y^2+y+1)(y-1)$$

Or thus:

Since $$a^3+b^3=(a+b)(a^2-ab+b^2),$$
we have $$(x-1)^3+y^3=(x-1+y)\{(x-1)^2-y(x-1)+y^2\},$$
and therefore $$\{(x-1)^3+y^3\}\div(x-1+y)=(x-1)^2-y(x-1)+y^2.$$

16. $$1-3+3-1\,)\,1-5+7-1-4+2\,(\,1-2-2$$
$$1-3+3-1$$
$$-2+4+0$$
$$-2+6-6+2$$
$$-2+6-6+2$$
$$-2+6-6+2$$

Ans. $x^2-2xy-2y^2$.

17. $a-b+2c \;)\; a^2+a(b-c)-2b^2+7bc-6c^2 \;(\; a+2b-3c$

$a^2-a(b-2c)$

$a(2b-3c)$

$a(2b-3c)+(2b-3c)(-b+2c)$

18. $a+b-c \;)\; a^2+a(3b+2c)+2b^2+bc-3c^2 \;(\; a+2b+3c$

$a^2+a(b-c)$

$a(2b+3c)$

$a(2b+3c)+(b-c)(2b+3c)$

20. $x^2+y^2-z^2+1 \;)\; x^4+2x^2y^2+y^4-z^4+2z^2-1 \;(\; x^2+y^2-z^2-1$

$x^4+x^2(y^2-z^2+1)$

$x^2(y^2+z^2-1)$

$x^2(y^2+z^2-1)+(y^2-z^2+1)(y^2+z^2-1)$

21. $(a-b)^3-c^3=\{(a-b)-c\}\{(a-b)^2+c(a-b)+c^2\}$.

Hence quotient required is $(a-b)^2+c(a-b)+c^2$.

22. $a+2b-c \;)\; a^3+\qquad\qquad 6abc+8b^3-c^3 \;(\; a^2-a(2b-c)+4b^2+2bc+c^2$

$a^3+a^2(2b-c)$

$-a^3(2b-c)$

$-a^2(2b-c)-a(2b-c)^2$

$a(4b^2+2bc+c^2)$

$a(4b^2+2bc+c^2)+(2b-c)(4b^2+2bc+c^2)$

23. $a^2-a(2b+3c)+4b^2-6bc+9c^2 \;)$

$a^3\qquad\qquad -18abc+8b^3+27c^3 \;(\; a+2b+3c$

$a^3-a^2(2b+3c)+a(4b^2-6bc+9c^2)$

$a^2(2b+3c)-a(4b^2+12bc+9c^2)+8b^3+27c^3$

$a^2(2b+3c)-a(2b+3c)^2+(2b+3c)(4b^2-6bc+9c^2)$

24. $3a-2b-3c \;)\; 27a^3\qquad\qquad -54abc-8b^3-27c^3 \;(\; 9a^2+3a(2b+3c)+4b^2-6bc+9c^2$

$27a^3-9a^2(2b+3c)$

$9a^2(2b+3c)$

$9a^2(2b+3c)-3a(2b+3c)^2$

$3a(4b^2-6bc+9c^2)$

$3a(4b^2-6bc+9c^2)-(2b+3c)(4b^2-6bc+9c^2)$

25. $ax-b \;)\; acx^3+(ad-bc)x^2-(ac+bd)x+bc \;(\; cx^2+dx-c$

acx^3-bcx^2

adx^2

adx^2-bdx

$-acx$

$-acx+bc$

26. $ax+2(b-c)y\,\big)\,2a^2x^2+abxy-2(b-c)(3b-4c)y^2\,\big(\,2ax-(3b-4c)y$

$2a^2x^2+4a(b-c)xy$

$-a(3b-4c)xy$

$-a(3b-4c)xy-2(b-c)(3b-4c)y^2$

27. Arranged in descending powers of a we have

$4+0-2+3\,\big)\,4-12+2+9-11+3\,\big(\,1-3+1$

$4+0-2+3$

$-12+4+6$

$-12+0+6-9$

$4+0-2+3$

$4+0-2+3$

Ans. $a^2-3ab+b^2$.

28. $(x^3+y^3)\div(x+y)=x^2-xy+y^2$.

Hence $\{(x+y)^3+z^3\}\div\{(x+y)+z\}=(x+y)^2-z(x+y)+z^2$.

29. $(x^3-y^3)\div(x-y)=x^2+xy+y^2$;

$\therefore\ \{(x+y)^3-8z^3\}\div\{(x+y)-2z\}=(x+y)^2+2z(x+y)+4z^2$.

IV. Page 56.

1. $a^4-16b^4=(a^2+4b^2)(a^2-4b^2)=(a^2+4b^2)(a+2b)(a-2b)$.

2. $16x^4-81a^4b^4=(4x^2+9a^2b^2)(4x^2-9a^2b^2)=(4x^2+9a^2b^2)(2x+3ab)(2x-3ab)$.

3. $16-(3a-2b)^2=\{4+(3a-2b)\}\{4-(3a-2b)\}$.

4. $4y^2-(2z-x)^2=\{2y+(2z-x)\}\{2y-(2z-x)\}$.

5. $5ax(4a^2x^2-9y^2)=5ax(2ax+3y)(2ax-3y)$.

6. $4a^2x^2(9x^4-y^4)=4a^2x^2(3x^2+y^2)(3x^2-y^2)$.

7. $8(a^4-b^4)=8(a^2+b^2)(a+b)(a-b)$.

8. $16(a^4-b^4)=16(a^2+b^2)(a+b)(a-b)$.

9. $(5x^2+2x-3+x^2-2x-3)(5x^2+2x-3-x^2+2x+3)$
$=24(x^2-1)(x^2+x)=24x(x+1)^2(x-1)$.

10. $(3x^2-4x-2+3x^2+4x-2)(3x^2-4x-2-3x^2-4x+2)=-16x(3x^2-2)$.

11. $4b^3(8a^3-b^6)=4b^3(2a-b^2)(4a^2+2ab^2+b^4)$.

12. $(a^2-2bc-2bc)\{(a^2-2bc)^2+2bc(a^2-2bc)+4b^2c^2\}$.

13. $(a-4)(a+2)$. **14.** $-(x-4)(x+3)$.

15. $(1-21x)(1+3x)$.

16. $-4(a-1)^2$.

17. $ab(a^2-4ab+3b^2)=ab(a-3b)(a-b)$.

18. $a^2b(a^2+5ab+4b^2)=a^2b(a+b)(a+4b)$.

19. $(b+c-a)(b+c-5a)$.

20. $\{3(a+b)-(c+d)\}^2$.

21. $(x^2-4)(x^2-25)=(x+2)(x-2)(x+5)(x-5)$.

22. $(25x^2-y^2)(4x^2-y^2)=(5x+y)(5x-y)(2x+y)(2x-y)$.

23. $(x^3-4y^2z^2)^2=(x+2yz)^2(x-2yz)^2$.

24. $a^2(9a^4-10a^2b^2+b^4)=a^2(9a^2-b^2)(a^2-b^2)$
$=a^2(3a+b)(3a-b)(a+b)(a-b)$.

25. $x^2-b^2-2a(x-b)=(x-b)(x+b-2a)$.

26. $x^2-a^2+2y(x-a)=(x-a)(x+a+2y)$.

27. $\{2ab+2cd+a^2+b^2-c^2-d^2\}\{2ab+2cd-a^2-b^2+c^2+d^2\}$
$=\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}$
$=(a+b+c-d)(a+b-c+d)(c+d-a+b)(c+d+a-b)$.

28. $(2xy-2ab+x^2+y^2-a^2-b^2)(2xy-2ab-x^2-y^2+a^2+b^2)$
$=\{(x+y)^2-(a+b)^2\}\{(a-b)^2-(x-y)^2\}$
$=(x+y+a+b)(x+y-a-b)(a-b+x-y)(a-b-x+y)$.

V. Pages 64—65.

1. $a(x^2-1)+x^3-x=(a+x)(x^2-1)=(a+x)(x+1)(x-1)$.

2. $a(c-d)+bc-bd=(a+b)(c-d)$.

3. $a(c^2-d^2)+bd^2-bc^2=(a-b)(c^2-d^2)=(a-b)(c+d)(c-d)$.

4. $a(cx^2+dxy)+bcxy+bdy^2=ax(cx+dy)+by(cx+dy)$
$=(ax+by)(cx+dy)$.

5. $a(cx^2+dx)+bcx^2+bd=ax(cx^2+d)+b(cx^2+d)=(ax+b)(cx^2+d)$.

6. $(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2=(a+b+c+d)(a+b-c-d)$
$+(a+b+c+d)(a+c-b-d)=(a+b+c+d)(a+b-c-d+a+c-b-d)$
$=2(a-d)(a+b+c+d)$.

7. $a^3(a+b)-b^3(a+b)=(a+b)(a^3-b^3)=(a+b)(a-b)(a^2+ab+b^2)$.

8. $a^3(a-b)-b^3(a-b)=(a-b)(a^3-b^3)=(a-b)^2(a^2+ab+b^2)$.

9. $a^2(b^2-1)-b^2+1=(b^2-1)(a^2-1)=(b+1)(b-1)(a+1)(a-1)$.

10. $x^2(y^2-z^2)-y^2z^2+z^4=(y^2-z^2)(x^2-z^2)=(y+z)(y-z)(x+z)(x-z)$.

11. $x^2(y^2z^2-z)-y^2z+1=x^2z(y^2z-1)-(y^2z-1)=(x^2z-1)(y^2z-1)$.

12. $x^3(x+y)+z^3(x+y)=(x+y)(x+z)(x^2-xz+z^2)$.

13. $x^2-y^2+z(x-y)=(x-y)(x+y+z)$.

14. $(x^2-9)(x^2+2)=(x+3)(x-3)(x^2+2)$.

15. $(x^2+1)^2-25x^2=(x^2+5x+1)(x^2-5x+1)$.

16. $(x^2+y^2)^2-16x^2y^2=(x^2+4xy+y^2)(x^2-4xy+y^2)$.

17. $(x^4+1)^2-x^4=(x^4-x^2+1)(x^4+x^2+1)$
$=(x^4-x^2+1)(x^2-x+1)(x^2+x+1)$.

18. $x^4-2(a^2+b^2)x^2+(a^2+b^2)^2-(a^2+b^2)^2+(a^2-b^2)^2=\{x^2-(a^2+b^2)\}^2-4a^2b^2$
$=(x^2-a^2-b^2-2ab)(x^2-a^2-b^2+2ab)=\{x^2-(a+b)^2\}\{x^2-(a-b)^2\}$
$=(x+a+b)(x-a-b)(x-a+b)(x+a-b)$.

19. $(x^2-2y^2z^2)^2$.

20. $\{x-(a+b)\}^2-(a+b)^2-ab(a-2)(b+2)=(x-a-b)^2-(ab+a-b)^2$
$=(x+ab-2b)(x-ab-2a)$.

21. $a(x+b)+x^3+bx^2=(a+x^2)(b+x)$.

22. $y^2(1-2x^2+x^4)+2y(1-x^4)+(1-2x^2+x^4)$
$=y^2(1-x^2)^2+2y(1-x^4)+(1-x^2)^2=(1-x^2)\{y^2(1-x^2)+2y(1+x^2)+(1-x^2)\}$
$=(1-x^2)\{(y+1)^2-(y-1)^2x^2\}=(1-x^2)\{y+1+x(y-1)\}\{y+1-x(y-1)\}$.

23. $x^2-2xz+z^2-y^2-4z^2+4yz=(x-z)^2-(y-2z)^2=(x+y-3z)(x-y+z)$.

24. Complete the square in y; then we have

$$=2\left[\left\{y-\frac{1}{4}(5x+a)\right\}^2-\frac{1}{16}(3x+3a)^2\right]$$
$$=2\left\{y-\frac{1}{4}(5x+a)+\frac{1}{4}(3x+3a)\right\}\left\{y-\frac{1}{4}(5x+a)-\frac{1}{4}(3x+3a)\right.$$
$$=(2y-x+a)(y-2x-a).$$

25. $a^2-2a(b+c)+(b+c)^2-(b+c)^2-3b^2-3c^2+10bc$
$=\{a-(b+c)\}^2-4(b-c)^2=\{a-(b+c)+2(b-c)\}\{a-(b+c)-2(b-c)\}$
$=(a+b-3c)(a-3b+c)$.

26. $2\left[a^2-\frac{a}{2}(7b+5)+\frac{1}{16}(7b+5)^2-\frac{1}{16}(7b+5)^2-11b^2+\frac{35}{2}b-\frac{3}{2}\right]$
$$=2\left[\left\{a-\frac{1}{4}(7b+5)\right\}^2-\frac{1}{16}\left\{225b^2-210b+49\right\}\right]$$
$$=2\left[\left\{a-\frac{1}{4}(7b+5)\right\}^2-\frac{1}{16}(15b-7)^2\right]$$
$$=(2a-11b+1)(a+2b-3).$$

27. $1-a^2x^2+bx^2(1-ax)=(1-ax)(1+ax+bx^2)$.

28. $1-2ax+a^2x^2-cx^2(1-ax)=(1-ax)(1-ax-cx^2)$.

29. $a^2(b-c)-a(b^2-c^2)+bc(b-c)=(b-c)\{a^2-a(b+c)+bc\}$
$=(b-c)(a-b)(a-c).$

30. $a^2(b+c)+a(b^2+c^2+2bc)+bc(b+c)=(b+c)\{a^2+a(b+c)+bc\}$
$=(b+c)(c+a)(a+b).$

31. $a^2(b+c)-a(b^2+c^2+2bc)+bc(b+c)=(b+c)\{a^2-a(b+c)+bc\}$
$=(b+c)(a-b)(a-c).$

32. $ax(x^2-xy+y^2)+x^3+y^3+by(x^2-xy+y^2)$
$=(x^2-xy+y^2)(ax+x+y+by).$

33. Complete the square in a: then we have

$$\frac{1}{by^2}\left[\left\{aby^2+\frac{1}{2}x(y^3+b^3)\right\}^2-\frac{1}{4}x^2(y^3-b^3)^2\right]=(ab+xy)(ay^2+b^2x).$$

34. $2x^3-4x^2y+2xy^2-z(x^2-2xy+y^2)=(x^2-2xy+y^2)(2x-z).$

35. $x^4yz+y^4zx+z^4xy-y^3z^3-z^3x^3-x^3y^3$
$=x^4yz-y^3z^3+y^4zx-x^3y^3+z^4xy-z^3x^3=yz(x^4-y^2z^2)+xy^3(yz-x^2)+xz^3(yz-x^2)$
$=(yz-x^2)\{xy^3+xz^3-yz(yz+x^2)\}=-(yz-x^2)(zx-y^2)(xy-z^2).$

36. $(x^2+x-12)(x^2+x-2)=(x+4)(x-3)(x+2)(x-1).$

37. $\{x^2+4x+8+2x\}\{x^2+4x+8+x\}=(x+4)(x+2)(x^2+5x+8).$

38. Taking the first and last factors together and also the middle factors, we have

$$(x^2+5x+4)(x^2+5x+6)-24=(x^2+5x)^2+10(x^2+5x)=x(x+5)(x^2+5x+10).$$

39. Taking the first and last factors together and also the middle factors, we have

$$(x^2+8x+7)(x^2+8x+15)+15=(x^2+8x)^2+22(x^2+8x)+120$$
$$=(x^2+8x+12)(x^2+8x+10)=(x+2)(x+6)(x^2+8x+10).$$

40. Taking the first and last factors together and also the middle factors, we have

$$4(x^2+17x+60)(x^2+16x+60)-3x^2$$
$$=4(x^2+60)^2+132x(x^2+60)+1085x^2$$
$$=\{2(x^2+60)+31x\}\{2(x^2+60)+35x\}$$
$$=(x+8)(2x+15)(2x^2+35x+120).$$

VI. Pages 73—75.

1. The given expression would vanish if $y-z=0$; and hence $y-z$ is a factor, and similarly $z-x$ and $x-y$ are factors. Hence, as the expression is of the *third* degree, it must be $L(y-z)(z-x)(x-y)$, where L is numerical. Comparing the coefficients of x^2y in the given expression and in

$$L(y-z)(z-x)(x-y),$$

we see that $L=3$.

2. The given expression would vanish if $y-z$ were zero. Hence $y-z$ is a factor, and similarly $z-x$ and $x-y$ are factors. Hence as the expression is of the fifth degree and is cyclically symmetrical, it must be

$$(y-z)(z-x)(x-y)(Ax^2+Ay^2+Az^2+Byz+Bzx+Bxy).....(i),$$

for $Ax^2+Ay^2+Az^2+Byz+Bzx+Bxy$ is the most general cyclically symmetrical expression of the second degree.

Equating the coefficients of x^4y in (i) and in the given expression, we have $A=5$.

Now put $x=0$, $y=1$, $z=2$; then

$$-1+32-1=(-1)\,2\,(-1)\,[5+20+2B],$$

whence $B=-5$.

Hence the given expression

$$=5(y-z)(z-x)(x-y)(x^2+y^2+z^2-yz-zx-xy).$$

3. $a^4(b^2-c^2)-a^2(b^4-c^4)+b^2c^2(b^2-c^2)=(b^2-c^2)\{a^4-a^2(b^2+c^2)+b^2c^2\}$

$$=-(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

4. It is easy to shew that $b-c$, $c-a$ and $a-b$ are factors. Hence as the given expression is of the *fourth* degree and is cyclically symmetrical there must be another cyclically symmetrical factor, and therefore the given expression must be

$$=L(b-c)(c-a)(a-b)(a+b+c).$$

Equating the coefficients of a^3b, we have $L=1$.

5. It is at once seen that $b-c$, $c-a$, $a-b$ are factors. Hence the given expression

$$=(b-c)(c-a)(a-b)\{L\Sigma a^3+M\Sigma a^2b+Nabc\}.....(i).$$

Now the coefficient of a^3b in the given expression is -1, and in (i) is $-L$; $\therefore L=1$.

The coefficient of a^4b^2 in the given expression is 0, and in (i) the coefficient is $L-M$; $\therefore M=L=1$.

To find N, put $a=1$, $b=2$, $c=3$ in the given expression and in (i); then

$$-1+64-3=2\{1+8+27+5+16+27+6N\},$$

whence $N=-9$.

Hence $\Sigma a(b-c)^5=(b-c)(c-a)(a-b)\{\Sigma a^3+\Sigma a^2b-9abc\}$.

6. We have

$$bc(b-c)+a(c^2-b^2)+a^2(b-c)=(b-c)\{bc-ab-ac+a^2\}$$

$$=-(b-c)(c-a)(a-b).$$

7. Arranging according to powers of a it becomes obvious that $b-c$ is a factor; then arranging according to powers of b it becomes obvious that $c-a$ is a factor; &c. Thus

$$\begin{aligned}
&b^3c^3(b-c)+a^3(c^4-b^4)+a^4(b^3-c^3)\\
&=(b-c)\{b^3c^3-a^3(b^3+b^2c+bc^2+c^3)+a^4(b^2+bc+c^2)\}\\
&=(b-c)\{b^3(c^3-a^3)+b^2a^3(a-c)+a^3bc(a-c)+c^2a^3(a-c)\}\\
&=(b-c)(c-a)\{b^3(c^2+ac+a^2)-b^2a^3-a^3bc-c^2a^3\}\\
&=(b-c)(c-a)\{c^3(b^3-a^3)+abc(b^2-a^2)+a^2b^2(b-a)\}\\
&=-(b-c)(c-a)(a-b)\{c^2(a^2+ab+b^2)+a^2bc+ab^2c+a^2b^2\}\\
&=-(b-c)(c-a)(a-b)\{b^3c^2+c^2a^2+a^2b^2+abc(a+b+c)\}.
\end{aligned}$$

8. $$\begin{aligned}
&a^4(b-c)-a(b^4-c^4)+bc(b^3-c^3)\\
&=(b-c)\{a^4-a(b^3+b^2c+bc^2+c^3)+bc(b^2+bc+c^2)\}\\
&=(b-c)\{b^3(c-a)+b^2c(c-a)+bc^2(c-a)-a(c^3-a^3)\}\\
&=(b-c)(c-a)\{b^3+b^2c+bc^2-a(c^2+ca+a^2)\}\\
&=-(b-c)(c-a)(a-b)\{a^2+b^2+c^2+bc+ca+ab\}.
\end{aligned}$$

9. $$\begin{aligned}
&(b-c)\{a^5-a(b^4+b^3c+b^2c^2+bc^3+c^4)+bc(b^3+b^2c+bc^2+c^3)\}\\
&=(b-c)(c-a)\{b^4+b^3c+b^2c^2+bc^3-a(a^3+a^2c+ac^2+c^3)\}\\
&=-(b-c)(c-a)(a-b)\{a^3+a^2b+ab^2+b^3+c(a^2+ab+b^2)\\
&\qquad\qquad +c^2a+c^2b+c^3\}\\
&=-(b-c)(c-a)(a-b)\{\Sigma a^3+\Sigma a^2b+abc\}.
\end{aligned}$$

10. The expression is evidently symmetrical and hence

$$=L\Sigma a^3+M\Sigma b^2c+Nabc.$$

The coefficient of a^3 is easily seen to be zero, and therefore $L=0$. The coefficient of a^2b is easily seen to be zero, and therefore $M=0$. The coefficient of abc is $6+6+6+6=24$. Hence the whole expression $=24abc$.

11. It is easy to see that abc is a factor. Hence, as the expression is symmetrical and of the fifth degree it

$$=abc\{L(a^2+b^2+c^2)+M(bc+ca+ab)\}.$$

Putting $a=b=c=1$, we have $243-1-1-1=3L+3M$; $\therefore L+M=80$.

Now put $a=1$, $b=1$, $c=2$;

then $960=12L+10M$, and $800=10L+10M$; whence $L=80$ and $M=0$.

Hence the whole expression $=80abc(a^2+b^2+c^2)$.

12. The coefficient of a^3 is 0; the coefficient of a^2b is 0; and the coefficient of abc is $2+2+2-2=4$. Hence expression $=4abc$.

13. The coefficient of a^3 is 0; the coefficient of a^2b is $1-1-1+1=0$; the coefficient of abc is 2. Hence the expression $=2abc$.

14. Arrange in powers of a. Coefficient of a^3 is $-1+1=0$; coefficient of a^2b is $1+1-1-1=0$; also coefficient of abc is $-2+2+2+2=4$. Hence expression $=4abc$.

15. The expression would vanish if $b-c$ were zero. Hence $b-c$ is a factor, and similarly $c-a$ and $a-b$ are factors. Hence the expression $=L(b-c)(c-a)(a-b)$.

Put $a=0$, $b=1$, $c=2$;

then $\quad -1.1.-1+2.-1.3+-1.3.1=L(-1.2.-1)$; $\therefore L=-4$.

16. We see at once that $y+z$, $z+x$, $x+y$ are factors. Hence the given expression $=L(y+z)(z+x)(x+y)$.

Put $x=y=z=1$; then $27-1-1-1=L.2.2.2$; $\therefore L=3$.

17. It can be seen at once that $y+z$, $z+x$ and $x+y$ are factors. Hence, as the expression is symmetrical and of the fifth degree, it must be equal to

$$(y+z)(z+x)(x+y)\{L(x^2+y^2+z^2)+M(yz+zx+xy)\}.$$

Put $x=y=z=1$; then $240=24L+24M$.

Put $x=0$, $y=1$, $z=1$; then $30=4L+2M$.

Whence $L=M=5$, so that given expression

$$=5(y+z)(z+x)(x+y)\{x^2+y^2+z^2+yz+zx+xy\}.$$

18. It can be seen at once that $b-c$, $c-a$ and $a-b$ are factors. Hence, as the expression is of the third degree, it must be equal to

$$L(b-c)(c-a)(a-b).$$

Put $a=1$, $b=2$, $c=3$; and we find $L=-1$.

19. It can be seen at once that $b-c$, $c-a$ and $a-b$ are factors. Hence, as the expression is cyclically symmetrical and of the fourth degree, it must be equal to

$$L(b-c)(c-a)(a-b)(a+b+c).$$

Equating coefficients of a^3b, we have

$$-L=-1+3; \quad \therefore L=-2.$$

20. It can be seen at once that $b-c$, $c-a$ and $a-b$ are factors. Hence, as the given expression is cyclically symmetrical and of the fifth degree, it must be equal to

$$(b-c)(c-a)(a-b)\{L(a^2+b^2+c^2)+M(bc+ca+ab)\}.$$

Equating the coefficients of a^4b and of a^3b^2 we have

$$3=-L, \quad \text{and} \quad 2=-M+L.$$

Hence the given expression

$$=-(b-c)(c-a)(a-b)\{3a^2+3b^2+3c^2+5bc+5ca+5ab\}.$$

21. As no factors are obvious arrange in terms of a. The coefficients can be easily picked out by inspection, and we have

$$a^2(b+c)+a(2bc+b^2+c^2)+b^2c+bc^2.$$

It is now obvious that $b+c$ is a factor, and the given expression is then seen to be equal to $(b+c)(c+a)(a+b)$.

22. It is easily seen that $b-c$, $c-a$ and $a-b$ are factors. Hence, as the given expression is a cyclically symmetrical expression of the *fifth* degree, it must be equal to

$$(b-c)(c-a)(a-b)\{L(a^2+b^2+c^2)+M(bc+ca+ab)\}.$$

Comparing coefficients of a^4b and of a^3b^2, we have

$$1=-L \text{ and } 1=L-M.$$

Hence the given expression

$$=-(b-c)(c-a)(a-b)\{\Sigma a^2+2\Sigma ab\}.$$

23. Arrange in powers of x; then we have

$$x^2(y+z)+x(y^2+3yz+z^2)+yz(y+z)\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

the factors of which are

$$x+y+z \text{ and } x(y+z)+yz.$$

If the factors of (i) cannot be seen by inspection, they can be found as in Art. 81, by writing the expression in the form

$$(y+z)\left[x^2+x\left(y+z+\frac{yz}{y+z}\right)+\frac{1}{4}\left(y+z+\frac{yz}{y+z}\right)^2-\frac{1}{4}\left(y+z-\frac{yz}{y+z}\right)^2\right]$$

$$=(y+z)\left\{x+\frac{1}{2}\left(y+z+\frac{yz}{y+z}\right)+\frac{1}{2}\left(y+z-\frac{yz}{y+z}\right)\right\}$$

$$\left\{x+\frac{1}{2}\left(y+z+\frac{yz}{y+z}\right)-\frac{1}{2}\left(y+z-\frac{yz}{y+z}\right)\right\}$$

$$=(x+y+z)(yz+zx+xy).$$

24. Arrange in powers of a; then

$$a^3(b+c)+a^2(2b^2+2c^2+4bc)+a(b+c)(3bc+b^2+c^2)+bc(b+c)^2$$
$$=(b+c)\{a^3+2a^2(b+c)+a(b^2+c^2+3bc)+bc(b+c)\}$$
$$=(b+c)\{b^2(c+a)+b(c^2+2a^2+3ac)+a^3+2a^2c+ac^2\}$$
$$=(b+c)(c+a)\{b^2+b(c+2a)+a(a+c)\}$$
$$=(b+c)(c+a)\{c(a+b)+(a+b)^2\}=(b+c)(c+a)(a+b)(a+b+c).$$

25. It is easily seen that x, y and z are factors. Hence, as the given expression is symmetrical and of the fourth degree, it must be equal to

$$Lxyz(x+y+z).$$

Put $x=y=z=1$; then $81-16-16-16+1+1+1=3L$.

Hence the given expression $=12xyz(x+y+z)$.

26. Arrange in terms of a: then

$$3a^2(b-c)-3a(b^2-c^2)+3bc(b-c)=-3(b-c)(c-a)(a-b).$$

27. It is easily seen that $b-c$, $c-a$, $a-b$, $a-d$, $b-d$ and $c-d$ are all factors. Hence, as the expression is of the *sixth* degree, it must be equal to

$$L(b-c)(c-a)(a-b)(a-d)(b-d)(c-d).$$

To find L, put $a=0$, $b=1$, $c=2$, $d=-1$; then we have $-192=12L$.

Hence the given expression

$$=-16(b-c)(c-a)(a-b)(a-d)(b-d)(c-d).$$

28. We have found in 25 that

$$(x+y+z)^4-(y+z)^4-(z+x)^4-(x+y)^4+x^4+y^4+z^4=12xyz(x+y+z).$$

Hence we have to prove that

$$(x+y+z)^{2n}-(y+z)^{2n}-(z+x)^{2n}-(x+y)^{2n}+x^{2n}+y^{2n}+z^{2n}$$

is divisible by x, y, z and $x+y+z$ for all positive integral values of n, and this follows at once from Art. 88.

29. The given expression would vanish if a were zero: hence a is a factor, and so also b and c are factors. The expression is therefore equal to

$$abc\{L(a^2+b^2+c^2)+M(bc+ca+ab)\}.$$

Put $a=b=c=1$; then we have $6=3L+3M$.

Put $a=1$, $b=1$, $c=-1$; then we have $2=-3L+M$. Hence $L=0$ and $M=2$. Thus the given expression $=2abc(bc+ca+ab)$.

30. Square the two sides of the identity

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b);$$

then $$\Sigma(b-c)^6+2\Sigma(c-a)^3(a-b)^3=9(b-c)^2(c-a)^2(a-b)^2.$$

Hence $$\Sigma(b-c)^6-9(b-c)^2(c-a)^2(a-b)^2=2\Sigma(a-b)^3(a-c)^3.$$

31. $(b+c)^3+(a+d)^3=(a+b+c+d)\{(b+c)^2-(b+c)(a+d)+(a+d)^2\}$,

$(c+a)^3+(b+d)^3=(a+b+c+d)\{(c+a)^2-(c+a)(b+d)+(b+d)^2\}$,

and $(a+b)^3+(c+d)^3=(a+b+c+d)\{(a+b)^2-(a+b)(c+d)+(c+d)^2\}$.

Hence the given expression $=3(a+b+c+d)(a^2+b^2+c^2+d^2)$.

32. We have $4(a^2+ab+b^2)^3-(2a^3+3a^2b-3ab^2-2b^3)^2$

$$=27a^4b^2+54a^3b^3+27a^2b^4=27a^2b^2(a+b)^2.$$

33. Since $\Sigma a^4-2\Sigma b^2c^2=-(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$, we have only to shew that $a\pm b\pm c$ are all factors of the given expression.

Now, if

$$a+b+c=0,\ b^2+c^2-a^2=-2bc,\ c^2+a^2-b^2=-2ca \text{ and } a^2+b^2-c^2=-2ab,$$

and the given expression would be $-8a^3b^3c^3(a+b+c)$. Thus the given expression would vanish if $a+b+c$ were zero. Hence $a+b+c$ is a factor of the expression, and similarly it can be shewn that $-a+b+c$, &c. are factors.

Or thus:—

The given expression is easily seen to be

$$-\Sigma a^{10}+3\Sigma a^8b^2-2\Sigma a^6b^4-6\Sigma a^6b^2c^2+6\Sigma a^4b^4c^4.$$

Now, if $\Sigma a^4 - 2\Sigma b^2c^2$ be one factor of the above it is easy to see that the other must be $-\Sigma a^6 + \lambda\Sigma a^4b^2 + \mu a^2b^2c^2$, where λ and μ are to be determined.

The necessary and sufficient conditions that the given expression may be

$$(\Sigma a^4 - 2\Sigma b^2c^2)(-\Sigma a^6 + \lambda\Sigma a^4b^2 + \mu a^2b^2c^2)$$

are that values of λ and μ can be found to satisfy simultaneously the equations

$$3 = \lambda + 2, \quad -2 = -1 - 2\lambda + \lambda, \quad -6 = 2 - 4\lambda + \mu \quad \text{and} \quad 6 = -2\lambda - 2\mu.$$

These equations are all satisfied by the values $\lambda = 1$, $\mu = -4$.

Hence the given expression

$$= (\Sigma a^4 - 2\Sigma b^2c^2)(-\Sigma a^6 + \Sigma a^4b^2 - 4a^2b^2c^2).$$

34. We have

$$(a^2-b^2)^2(c^2-d^2)^2 + 16a^2b^2c^2d^2 + 4c^2d^2(a^2-b^2)^2 + 4a^2b^2(c^2-d^2)^2$$
$$= (c^2-d^2)^2\{(a^2-b^2)^2 + 4a^2b^2\} + 4c^2d^2\{(a^2-b^2)^2 + 4a^2b^2\} = (c^2+d^2)^2(a^2+b^2)^2.$$

35. The expression on the left

$$(y^2-z^2)\{1 + x(y+z) + x^2yz\} + \&c.$$
$$= x(y+z)(y^2-z^2) + y(z+x)(z^2-x^2) + z(x+y)(x^2-y^2)$$
$$+ xyz\{x(y^2-z^2) + y(z^2-x^2) + z(x^2-y^2)\}.$$

And it is easily seen that

$$\Sigma x(y+z)(y^2-z^2) = (y-z)(z-x)(x-y)(x+y+z),$$

and

$$\Sigma x(y^2-z^2) = (y-z)(z-x)(x-y).$$

36. It is at once seen that $b-c$, $c-a$, $a-b$, $a-d$, $b-d$ and $c-d$ are all factors. Hence, as the given expression is of the sixth degree, it must be

$$L(b-c)(c-a)(a-b)(a-d)(b-d)(c-d).$$

The coefficient of a^3b^2b in the above is $-L$, and in the given expression is -1; hence $L=1$.

37. It is at once seen that $b-c$, $c-a$, $a-b$, $a-d$, $b-d$ and $c-d$ are all factors of the given expression which is of the ninth degree. Hence it must be equal to

$$(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)\{\Sigma La^3 + \Sigma Ma^2b + \Sigma Nabc\}.$$

Now, in the given expression, the coefficient of a^6b^2c is zero, and therefore $L=0$; the coefficient of a^5b^3c is zero, and therefore $M=0$; thus there are no terms of the form a^3 or a^2b, so that the given expression must be

$$(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)\{N_1bcd + N_2cda + N_3dab + N_4abc\}.$$

Comparing the coefficients of $a^4b^3c^2$ in the given expression and in that last written we have $N_4 = -1$, and we find in a similar manner that

$$N_1 = N_2 = N_3 = -1.$$

Hence we have

$$-(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)(bcd + cda + dab + abc).$$

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4. The difference of the given expressions

$$= 2x^3 - 3x^2y + xy^2 = x\,(2x - y)\,(x - y).$$

And, from Art. 88, it is obvious that $2x - y$ is a factor of $2x^3 + 3x^2y - y^3$ and that x and $x - y$ are not factors. Hence $2x - y$ is the H. C. F. required.

5. The expressions are respectively

$$x^2 - (2y - 3z)^2 \text{ and } (x + z)^2 - (2y - 2z)^2,$$

that is $(x + 2y - 3z)\,(x - 2y + 3z)$ and $(x + 2y - z)\,(x - 2y + 3z)$.

Hence $x - 2y + 3z$ is the H. C. F. required.

6.

$$\begin{array}{l}
1+0+0-3+20 \;)\; 5-3+0+\;0+\;64 \;(\; 5 \\
\qquad\qquad\quad 5+0+0-15+100 \\
\qquad\qquad\quad \overline{-3+0+15-\;36} \\
\qquad\qquad\quad -3 \;|\; 1+0-\;5+\;12 \;)\; 1+0+0-\;3+20 \;(\; 1 \\
\qquad\qquad\qquad\qquad\qquad\quad 1+0-5+12 \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{5-15+20} \\
\qquad\qquad\qquad\qquad\qquad\quad 5 \;|\; 1-\;3+\;4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 1-3+4 \;)\; 1+0-5+12 \;(\; 1+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1-3+4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{3-9+12} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3-9+12 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad\qquad}
\end{array}$$

Ans. $b^2 - 3ba + 4a^2$.

7.

$$\begin{array}{l}
1-1+1+14 \;)\; 4+3-\;9+\;2 \;(\; 4 \\
\qquad\qquad\quad 4-4+\;4+56 \\
\qquad\qquad\quad \overline{7-13-54} \;)\; 7-\;7+\;\;7+\;\;98 \;(\; 1 \\
\qquad\qquad\qquad\qquad\quad 7-13-\;54 \\
\qquad\qquad\qquad\qquad\quad \overline{\qquad 6+\;61+\;\;98} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 7 \\
\qquad\qquad\qquad\qquad\quad \overline{42+427+\;686} \;(\; 6 \\
\qquad\qquad\qquad\qquad\quad 42-\;78-\;324 \\
\qquad\qquad\qquad\qquad\quad \overline{505 \;|\; 505+1010} \\
\qquad\qquad\qquad\qquad\qquad\qquad 1+\;\;2 \;)\; 7-13-54 \;(\; 7-27 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; 7+14 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; \overline{-27-54} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; -27-54 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; \overline{\qquad\qquad}
\end{array}$$

Ans. $a + 2b$.

8.

$$\begin{array}{r|l} 2+1-9+8-2 & 2-7+11-8+2 \\ & 2+1-9+8-2 \\ \hline & -4 \\ & -8+20-16+4 \\ & 2-5+4-1 \end{array}$$

$$\begin{array}{r|l} 2-5+4-1 & 2+1-9+8-2 \\ & 2-5+4-1 \\ \hline & 6-13+9-2 \\ & 6-15+12-3 \\ \hline & 2-3+1 \end{array}$$

$$\begin{array}{r|l} 2-3+1 & 2-5+4-1 \\ & 2-3+1 \\ \hline & -2+3-1 \\ & -2+3-1 \\ \hline & \end{array}$$

Ans. $2x^2-3x+1$.

9. $a^4+a^2x^2+9ax^3-11x^4$

$$\begin{array}{r|l} a^4+2a^2x^2+10ax^3-13x^4 & 1 \\ a^4+a^2x^2+9ax^3-11x^4 & \\ \hline a^2x^2+ax^3-2x^4 = x^2(a+2x)(a-x). & \end{array}$$

Now, from Art. 88, it is at once seen that $x-a$ is a factor, and that x^2 and $2x+a$ are not factors of the first of the given expressions; hence the H. C. F. required is $x-a$.

10.

$$\begin{array}{r|l} 1+1-9-3+18 & 1+0+0+6-49+42 \\ & 1+1-9-3+18 \\ \hline & -1+9+9-67+42 \\ & -1-1+9+3-18 \\ \hline & 10 \\ & 10+0-70+60 \\ & 1+0-7+6 \end{array}$$

$$\begin{array}{r|l} 1+0-7+6 & 1+1-9-3+18 \\ & 1+0-7+6 \\ \hline & 1-2-9 \\ & 1+0-7+6 \\ \hline & -2 \\ & -2-2+12 \\ & 1+1-6 \end{array}$$

$$\begin{array}{r|l} 1+1-6 & 1+0-7+6 \\ & 1+1-6 \\ \hline & -1-1+6 \\ & -1-1+6 \\ \hline & \end{array}$$

Ans. x^2+x-6.

11.

$$\begin{array}{r|l} 1-2+5-4+3 & 2-1+6+2+3 \\ & 1-2+5-4+3 \\ \hline & 1+1+1+6 \end{array}$$

$$\begin{array}{r|l} 1+1+1+6 & 1-2+5-4+3 \\ & 1+1+1+6 \\ \hline & -3+4-10+3 \\ & -3-3-3-18 \\ \hline & 7 \\ & 7-7+21 \\ & 1-1+3 \end{array}$$

$$\begin{array}{r|l} 1-1+3 & 1+1+1+6 \\ & 1-1+3 \\ \hline & 2-2+6 \\ & 2-2+6 \\ \hline \end{array} \quad \begin{array}{l} \big(1+2 \\ \\ \\ \\ \\ \end{array}$$

Ans. x^2-x+3.

12. $$\begin{array}{r|l} 1+0+3+6+35 & 1+2-5+26+21 \\ & 1+0+3+\ 6+35 \\ \hline & 2\,|\,2-8+20-14 \\ & \\ 1+4+10-\ 7 & 1+0+\ 3+\ 6+35 \\ & 1-4+10-\ 7 \\ \hline & 4-\ 7+13+35 \\ & 4-16+40-28 \\ \hline & 9\,|\,9-27+63 \\ & 1-\ 3+\ 7 \\ \hline \end{array} \quad \begin{array}{l} \big(1 \\ \\ \\ \\ \\ \\ \\ \\ \big(1+4 \\ \\ \\ \\ \\ \end{array}$$

$$\begin{array}{r|l} 1-3+7 & 1-4+10-7 \\ & 1-3+\ 7 \\ \hline & -1+\ 3-7 \\ & -1+\ 3-7 \\ \hline \end{array} \quad \begin{array}{l} \big(1-1 \\ \\ \\ \\ \\ \end{array}$$

Ans. x^2-3x+7.

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12. By Art. 98, any common factor of the two given expressions is also a factor of

$$a'(ax^2+bx+c)-a(a'x^2+b'x+c),$$

and of

$$c'(ax^2+bx+c)-c(a'x^2+b'x+c'),$$

that is of

$$x(a'b-ab')+a'c-ac'$$

and

$$x^2(ac'-a'c)+x(bc'-b'c).$$

Hence, if the given expression have a common factor of the form $x+1$, this factor must divide both

$$x(a'b-ab')+a'c-ac'$$

and

$$x(ac'-a'c)+bc'-b'c,$$

whence it follows that

$$\frac{a'b-ab'}{ac'-a'c}=\frac{a'c-ac'}{bc'-b'c}.$$

[See also Ex. 3, Art. 153.]

13. By Art. 98, any common factor of the two given expressions is also a factor of

$$a'(ax^3+bx^2+cx+d)-a(a'x^3+b'x^2+c'x+d')$$

and of $$d'(ax^3+bx^2+cx+d)-d(a'x^3+b'x^2+c'x+d'),$$

that is of $$(ba'-b'a)x^2+(ca'-c'a)x+da'-d'a$$

and of $$(ad'-a'd)x^2+(bd'-b'd)x^2+(cd'-c'd)x.$$

Hence, if the given expressions have as a common factor a quadratic expression in x, that common factor can only differ from

$$(ba'-b'a)x^2+(ca'-c'a)x+(da'-d'a) \quad \text{.....(i),}$$

or from

$$(ad'-a'd)x^2+(bd'-b'd)x+(cd'-c'd) \quad \text{..... .....(ii)}$$

by a factor which does not contain x, and therefore the expressions (i) and (ii) can only differ by such a factor, whence it follows that

$$\frac{ba'-b'a}{ad'-a'd}=\frac{ca'-c'a}{bd'-b'd}=\frac{da'-d'a}{cd'-c'd}.$$

14. Any common factor must also be a factor of

$$c'(ax^2+bx+c)-c(a'x^2+b'x+c'),$$

and therefore of

$$(ac'-a'c)x^2+bc'-b'c \quad \text{.....(i).}$$

Again any common factor of the given expressions must be a factor of

$$a'(ax^2+bx+c)-a(a'x^2+b'x+c'),$$

and therefore of

$$(ba'-b'a)x+ca'-c'a \quad \text{.....(ii).}$$

Hence $x+(ca'-c'a)/(ba'-b'a)$ must be a common factor, and the condition that the above should be a factor of (i) is, from Art. 88,

$$(ca'-c'a)^2-(bc'-b'c)(ba'-b'a)^2=0.$$

15. From Art. 105, we have

$$l_1g_1=bc,\ l_2g_2=ca \text{ and } l_3g_3=ab;$$

$$\therefore\ l_1l_2l_3g_1g_2g_3=(abc)^2.$$

16. Let $(BC)=(ABC)\alpha$, $(CA)=(ABC)\beta$ and $(AB)=(ABC)\gamma$; then α, β, γ have by supposition no factors in common.

We must also have the following relations:

$$A=(ABC).\beta.\gamma.x,$$

$$B=(ABC).\gamma.\alpha.y,$$

and $$C=(ABC).\alpha.\beta.z,$$

where x, y, z have no common factors.

Now the L. C. M. of A, B and C is obviously

$$= (ABC)\,\alpha\beta\gamma xyz$$

$$= \frac{(ABC)^4\,\alpha^2\beta^2\gamma^2 xyz}{\{(ABC)\alpha\}\,\{(ABC)\beta\}\,\{(ABC)\gamma\}} = \frac{ABC(ABC)}{(BC)(CA)(AB)}.$$

IX. Pages 99—104.

1. $$\frac{30a^2b^3c^5x^2y^4z^8}{36a^5bc^2x^5yz^6} = \frac{5b^2c^3y^3z^2 \times 6a^2bc^2x^2yz^6}{6a^3x^3 \times 6a^2bc^2x^2yz^6} = \frac{5b^2c^3y^3z^2}{6a^3x^3}.$$

2. $$\frac{3a^7b^2c^{10}x^3yz^4}{a^6c^4x^3y^6} = \frac{3ab^2c^6x^5z^4 \times a^6c^4x^3y}{y^5 \times a^6c^4x^3y} = \frac{3ab^2c^6x^5z^4}{y^5}.$$

3. $$\frac{a^2-8ab+7b^2}{a^2-3ab-28b^2} = \frac{(a-7b)(a-b)}{(a-7b)(a+4b)} = \frac{a-b}{a+4b}.$$

4. $$\frac{7x^4y^4-8x^2y^2+1}{28x^4y^4+3x^2y^2-1} = \frac{(7x^2y^2-1)(x^2y^2-1)}{(7x^2y^2-1)(4x^2y^2+1)} = \frac{x^2y^2-1}{4x^2y^2+1}.$$

5. $$\frac{(x^3-y^3)(x+y)}{(x^3+y^3)(x-y)} = \frac{(x^2+xy+y^2)(x-y)(x+y)}{(x^2-xy+y^2)(x-y)(x+y)} = \frac{x^2+xy+y^2}{x^2-xy+y^2}.$$

6. $$\frac{(x^6-y^6)(x-y)}{(x^3-y^3)(x^4-y^4)} = \frac{(x^3-y^3)(x^2-xy+y^2)(x+y)(x-y)}{(x^3-y^3)(x^2+y^2)(x^2-y^2)} = \frac{x^2-xy+y^2}{x^2+y^2}.$$

7. Dividing the numerator and the denominator by their H. C. F. $(x+1)^2$, we have $\dfrac{2x-1}{x^2+1}$.

8. Dividing the numerator and the denominator by their H. C. F. x^2+x+1, we have $\dfrac{(x-1)^2}{x^2-3x+1}$.

9. Dividing the numerator and the denominator by their H. C. F. x^2+xy-y^2, we have $\dfrac{2x+3y}{3x^2-y^2}$.

10. Dividing the numerator and the denominator by their H. C. F. $6x^2-3x+2$, we have $\dfrac{9x^3-3x-2}{6x^3+3x^2-1}$.

11. The fraction

$$= \frac{(a+b)(a+b+c)(a+b-c)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} = \frac{a+b}{(-a+b+c)(a-b+c)}.$$

12. The numerator $= -(y-z)(z-x)(x-y)(yz+zx+xy)$. [Art. 93, Ex. 3].
The denominator $= -(y-z)(z-x)(x-y)$.
Hence the fraction $= yz+zx+xy$.

13. The numerator $= -(y-z)(z-x)(x-y)(x^2+y^2+z^2+yz+zx+xy)$. [See Examples VI. 8.]
The denominator $= 2(x^2+y^2+z^2+yz+zx+xy)$.
Hence the fraction $= -\frac{1}{2}(y-z)(z-x)(x-y)$.

14. $$\frac{a(b-c)(c-d)-c(d-a)(a-b)}{b(c-d)(d-a)-d(a-b)(b-c)} = \frac{a^2c-ac^2-bd(a-c)}{b^2d-bd^2-ac(b-d)}$$
$$= \frac{(ac-bd)(a-c)}{(bd-ac)(b-d)} = \frac{a-c}{d-b}.$$

15. The numerator $= -(y-z)(z-x)(x-y)(yz+zx+xy)$.
The denominator $= -(y-z)(z-x)(x-y)(x+y+z)$.
Hence the fraction $= \frac{yz+zx+xy}{x+y+z}$.

16. The expression $= \frac{2a(a-b)+2b(a+b)-(a^2+b^2)}{a^2-b^2} = \frac{a^2+b^2}{a^2-b^2}$.

17. The expression
$$= \frac{(3-x)(1+3x)-(3+x)(1-3x)+(1-16x)}{1-9x^2} = \frac{1}{1-9x^2}.$$

18. We have $\frac{x(x-2y)+y(x+2y)-(x-y)^2}{x^2-4y^2} = \frac{xy+y^2}{x^2-4y^2}$.

19. We have
$$\frac{(x-2a)^2+(x+2a)^2+8ax}{x^2-4a^2} = \frac{2(x+2a)^2}{x^2-4a^2} = \frac{2x+4a}{x-2a}.$$

20. Whole expression
$$= \frac{(x+8)-(x+2)}{(x+2)(x+8)} + \frac{-3(x+6)+3(x+4)}{(x+4)(x+6)} = \frac{6}{(x+2)(x+8)} - \frac{6}{(x+4)(x+6)}$$
$$= \frac{6(x+4)(x+6)-6(x+2)(x+8)}{(x+2)(x+4)(x+6)(x+8)} = \frac{48}{(x+2)(x+4)(x+6)(x+8)}.$$

21. We have
$$\frac{6a}{(x+a)(x+7a)} - \frac{6a}{(x+3a)(x+5a)} = \frac{48a^3}{(x+a)(x+3a)(x+5a)(x+7a)}.$$

22. We have

$$\frac{2x}{x^2-4a^2}-\frac{8x}{x^2-a^2}+\frac{6}{x}=\frac{2x^2(x^2-a^2)-8x^2(x^2-4a^2)+6(x^2-a^2)(x^2-4a^2)}{x(x^2-a^2)(x^2-4a^2)}$$

$$=\frac{24a^4}{x(x^2-a^2)(x^2-4a^2)}.$$

23. We have $$\frac{(x-y)-2(x-2y)+(x-3y)}{(x-y)(x-2y)(x-3y)}=0.$$

24. We have $$\frac{-a(b-c)-b(c-a)-c(a-b)}{(b-c)(c-a)(a-b)}=0.$$

25. We have $$\frac{-a^2(b-c)-b^2(c-a)-c^2(a-b)}{(b-c)(c-a)(a-b)}=\frac{(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}=1.$$

26. We have

$$\frac{\Sigma(1+ab)(1+ac)(c-b)}{(b-c)(c-a)(a-b)}=\frac{\Sigma(c-b)+\Sigma a(c^2-b^2)+abc\Sigma a(c-b)}{(b-c)(c-a)(a-b)}$$

$$=\frac{\Sigma a(c^2-b^2)}{(b-c)(c-a)(a-b)}=-1.$$

27. We have

$$\frac{\Sigma bc(a+d)(c-b)}{(b-c)(c-a)(a-b)}=\frac{\Sigma abc(c-b)+d\Sigma bc(c-b)}{(b-c)(c-a)(a-b)}=\frac{d(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}=d.$$

28. We have $$\frac{(x^2-yz)(y+z)+(y^2-zx)(z+x)+(z^2-xy)(x+y)}{(y+z)(z+x)(x+y)}=0.$$

29. We have $$\frac{\Sigma(y-x)(z-x)(-2x+y+z)}{(-2x+y+z)(x-2y+z)(x+y-2z)};$$

or, putting $y-z=a$, $z-x=b$ and $x-y=c$, we have

$$\frac{\Sigma-cb(b-c)}{(b-c)(c-a)(a-b)}=1.$$

30. The denominator

$$=\frac{1}{2}\left[\frac{2x}{x-a}-1+\frac{2x}{x-b}-1+\frac{2x}{x-c}-1-3\left\{\frac{2x^3+2x\Sigma bc}{(x-a)(x-b)(x-c)}-1\right\}\right]$$

$$=\frac{1}{2}\left[\frac{x+a}{x-a}+\frac{x+b}{x-b}+\frac{x+c}{x-c}-3\frac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)}\right],$$

since $2x^3+2x\Sigma bc-(x-a)(x-b)(x-c)=x^3+x^2\Sigma a+x\Sigma bc+abc$
$=(x+a)(x+b)(x+c)$.

Thus the denominator is one-half the numerator.

31. We have

$$\frac{-\Sigma a^3(b-c)}{(b-c)(c-a)(a-b)} = \frac{(b-c)(c-a)(a-b)(a+b+c)}{(b-c)(c-a)(a-b)} = a+b+c.$$

32. We have

$$\frac{-\Sigma a^4(b-c)}{(b-c)(c-a)(a-b)} = \frac{(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)}{(b-c)(c-a)(a-b)} = \Sigma a^2+\Sigma bc.$$

33. We have

$$\frac{-\Sigma a^2(a+b)(a+c)(b-c)}{(b-c)(c-a)(a-b)} = \frac{(b-c)(c-a)(a-b)(a+b+c)^2}{(b-c)(c-a)(a-b)} \quad [\text{VI. 22}].$$

34. The numerator $= \dfrac{\Sigma a^3(c-b)}{abc} = \dfrac{(b-c)(c-a)(a-b)(a+b+c)}{abc}$.

The denominator $= \dfrac{\Sigma a^2(c-b)}{abc} = \dfrac{(b-c)(c-a)(a-b)}{abc}$.

Hence the fraction $= a+b+c$.

35. The fraction

$$= \{(-a+b+c)+(a-b+c)+(a+b-c)\}/\{(-a+b+c)(a-b+c)(a+b-c)\}$$
$$= (a+b+c)/\{(-a+b+c)(a-b+c)(a+b-c)\}.$$

36. The given expression

$$= \frac{b-c}{(a+b-c)(a-b+c)} + \frac{c-a}{(b+c-a)(b-c+a)} + \frac{a-b}{(c+a-b)(c-a+b)}$$
$$= \frac{(b-c)(-a+b+c)+(c-a)(a-b+c)+(a-b)(a+b-c)}{(a+b-c)(a-b+c)(-a+b+c)} = 0.$$

37. $$16+\left\{\frac{x+a}{x-a}+\frac{x-a}{x+a}-2\frac{x^2-a^2}{x^2+a^2}\right\}^2 = 16+\left\{\frac{2(x^2+a^2)}{x^2-a^2}-\frac{2(x^2-a^2)}{x^2+a^2}\right\}^2$$
$$= 16+4\left(\frac{4a^2x^2}{x^4-a^4}\right)^2 = 16\left(\frac{x^4+a^4}{x^4-a^4}\right)^2.$$

38. $$\frac{a+b}{ax+by}+\frac{a-b}{ax-by}+2\,\frac{a^2x+b^2y}{a^2x^2+b^2y^2} = \frac{2(a^2x-b^2y)}{a^2x^2-b^2y^2}+\frac{2(a^2x+b^2y)}{a^2x^2+b^2y^2}$$
$$= 4\,\frac{a^4x^3-b^4y^3}{a^4x^4-b^4y^4}.$$

39. (i) We have $\dfrac{\Sigma a^2(c-b)(1+bx)(1+cx)}{(b-c)(c-a)(a-b)(1+ax)(1+bx)(1+cx)}$.

Now, in the numerator, the coefficient of x^2 is $abc\Sigma a(c-b)=0$, the coefficient of x is $\Sigma a^2(c^2-b^2)=0$, and the terms which do not contain x are $\Sigma a^2(c-b)=(b-c)(c-a)(a-b)$.

Hence the whole expression is equal to $\frac{1}{(1+ax)(1+bx)(1+cx)}$.

(ii) We have $\frac{\Sigma a(c-b)(1+bx)(1+cx)}{(b-c)(c-a)(a-b)(1+ax)(1+bx)(1+cx)}$.

In the numerator the coefficient of x^2 is $abc\Sigma(c-b)=0$, the coefficient of x is $\Sigma a(c^2-b^2)=-(b-c)(c-a)(a-b)$, and the terms which do not contain x are $\Sigma a(c-b)=0$. Hence the whole expression $=\frac{-x}{(1+ax)(1+bx)(1+cx)}$.

(iii) We have $\frac{\Sigma(c-b)(1+bx)(1+cx)}{(b-c)(c-a)(a-b)(1+ax)(1+bx)(1+cx)}$.

The coefficient of x^2 is $\Sigma bc(c-b)=(b-c)(c-a)(a-b)$, the coefficient of x is $\Sigma(c^2-b^2)=0$, and the terms which do not contain x are $\Sigma(c-b)=0$. Hence the whole expression $=\frac{x^2}{(1+ax)(1+bx)(1+cx)}$.

40. We have $\frac{\Sigma(a+p)(a+q)(x+b)(x+c)(c-b)}{(b-c)(c-a)(a-b)(x+a)(x+b)(x+c)}$.

The numerator

$$=x^2\Sigma(c-b)(a+p)(a+q)+x\Sigma(c^2-b^2)(a+p)(a+q)+\Sigma bc(c-b)(a+p)(a+q).$$

Now $\Sigma bc(c-b)(a+p)(a+q)=abc\Sigma a(c-b)+abc(p+q)\Sigma(c-b)$
$+pq\Sigma bc(c-b)=pq(b-c)(c-a)(a-b)$,

$\Sigma(c^2-b^2)(a+p)(a+q)=\Sigma a^2(c^2-b^2)+(p+q)\Sigma a(c^2-b^2)+pq\Sigma(c^2-b^2)$
$=-(p+q)(b-c)(c-a)(a-b)$,

and $\Sigma(c-b)(a+p)(a+q)=\Sigma a^2(c-b)+(p+q)\Sigma a(c-b)+pq\Sigma(c-b)$
$=(b-c)(c-a)(a-b)$.

Hence the numerator

$$=(b-c)(c-a)(a-b)\{x^2-(p+q)x+pq\}.$$

The whole expression is therefore equal to $\frac{(x-p)(x-q)}{(x+a)(x+b)(x+c)}$.

41. We have $\frac{\Sigma a(b+c-a)(c-b)}{(b-c)(c-a)(a-b)}=\frac{2\Sigma a^2(b-c)}{(b-c)(c-a)(a-b)}=-2$.

42. We have $\frac{\Sigma(a-b+c)(a+b-c)(c-b)}{(b-c)(c-a)(a-b)}=\frac{\Sigma a^2(c-b)+\Sigma(b-c)^3}{(b-c)(c-a)(a-b)}$
$=\frac{(b-c)(c-a)(a-b)+3(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}=4$.

43. We have $\frac{\Sigma a(b+c)\{a^2-(b-c)^2\}}{(-a+b+c)(a-b+c)(a+b-c)}$.

The numerator is a symmetrical expression of the fourth degree, and therefore $=A\Sigma a^4+B\Sigma a^3b+C\Sigma a^2b^2+D\Sigma a^2bc$, where A, B, C, D are numerical. But, by inspection, the coefficients of a^4, a^3b and a^2b^2 are all zero, and the coefficient of a^2bc is 2. Hence the numerator $=2\Sigma a^2bc=2abc(a+b+c)$.

44. $\left(m+\frac{1}{m}\right)^2+\left(n+\frac{1}{n}\right)^2+\left(mn+\frac{1}{mn}\right)^2$

$$=m^2+n^2+m^2n^2+\frac{1}{m^2}+\frac{1}{n^2}+\frac{1}{m^2n^2}+6\,;$$

also $\left(m+\frac{1}{m}\right)\left(n+\frac{1}{n}\right)\left(mn+\frac{1}{mn}\right)=m^2+n^2+m^2n^2+\frac{1}{m^2}+\frac{1}{n^2}+\frac{1}{m^2n^2}+2.$

45. $\left(\frac{2bc}{b+c}-b\right)\div\left(\frac{1}{c}+\frac{1}{b-2c}\right)=\frac{b\,(c-b)}{b+c}\times\frac{c\,(b-2c)}{b-c}=\frac{-bc\,(b-2c)}{b+c}$.

The other term $=\frac{-bc\,(c-2b)}{b+c}$.

Hence the whole expression $=bc$.

46. We have to shew that

$$(b-c)\,(1+ca)\,(1+ab)+(c-a)\,(1+ab)\,(1+bc)+(a-b)\,(1+bc)\,(1+ca)$$
$$=(b-c)\,(c-a)\,(a-b).$$

The expression on the left

$$=\Sigma\,(b-c)+\Sigma a\,(b^2-c^2)+abc\,\Sigma a\,(b-c)=\Sigma a\,(b^2-c^2)=(b-c)\,(c-a)\,(a-b).$$

47. We have $\frac{(yz+zx+xy)^2}{xyz}-\frac{y^2z^2+z^2x^2+x^2y^2}{xyz}$

$$=2\,(x^2yz+y^2zx+z^2xy)/xyz=2\,(x+y+z).$$

48. Let $\frac{y+z}{b-c}=\frac{z+x}{c-a}=\frac{x+y}{a-b}=\lambda$;

then $y+z=\lambda\,(b-c)$, $z+x=\lambda\,(c-a)$ and $x+y=\lambda\,(a-b)$;

$\therefore$, by addition, $x+y+z=0$;

$$\therefore\ -x=\lambda\,(b-c),\ -y=\lambda\,(c-a),\ \text{and}\ -z=\lambda\,(a-b);$$
$$\therefore\ x^2+y^2+z^2=\lambda^2\,\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$

49. Let $\frac{x}{y}=\frac{a}{b}=\lambda$; then $x=\lambda y$ and $a=\lambda b$; hence

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}=\frac{\lambda^2\,(y^2+b^2)}{\lambda\,(y+b)}+\frac{y^2+b^2}{y+b}=(\lambda+1)\frac{y^2+b^2}{y+b}.$$

Also $$\frac{(x+y)^2+(a+b)^2}{x+y+a+b}=\frac{(1+\lambda)^2\,(y^2+b^2)}{(1+\lambda)\,(y+b)}.$$

50. Since $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=\lambda$ suppose, we have

$$x=\lambda\,(b+c-a),\ y=\lambda\,(c+a-b)\ \text{and}\ z=\lambda\,(a+b-c).$$

Hence $\Sigma\,(b-c)\,x=\lambda\,\{(b-c)\,(b+c-a)+(c-a)\,(c+a-b)+(a-b)\,(a+b-c)\}$
$$=\lambda\,\{\Sigma\,(b^2-c^2)-\Sigma a\,(b-c)\}=0.$$

51. We have $\frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\lambda$ suppose;

$$\therefore\ bz-cy=(b-c)\lambda \text{ and } cx-az=(c-a)\lambda.$$

Hence $$a(bz-cy)+b(cx-az)=\lambda\{a(b-c)+b(c-a)\};$$

$\therefore\ c(bx-ay)=\lambda c(b-a)$. Hence, *unless* $c=0$, we have $(ay-bx)=\lambda(a-b)$. Hence also, by addition, $bz-cy+cx-az+ay-bx=\lambda(b-c+c-a+a-b)=0$, and therefore $\Sigma x(b-c)=0$.

52. Multiply by $(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$; then we have to shew that

$$-a^4(b-c)(b-d)(c-d)-b^4(c-a)(c-d)(a-d)$$
$$-c^4(a-b)(a-d)(b-d)-d^4(b-c)(c-a)(a-b)$$
$$=(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)(a+b+c+d).$$

Now, it follows at once from Art. 88 that $b-c$, $c-a$, $a-b$, $a-d$, $b-d$, $c-d$ are all factors of the expression on the left. Hence, as that expression is cyclically symmetrical and of the seventh degree, it must be equal to

$$L(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)(a+b+c+d),$$

and comparing the coefficients of a^4b^2c we have $-1=-L$.

53. Multiply by

$$\{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)(a_2-a_3)(a_2-a_4)\ldots(a_2-a_n)$$
$$(a_3-a_4)\ldots(a_3-a_n)\ldots\ldots(a_{n-1}-a_n)\};$$

then the expression becomes

$$\Sigma a_1^r\{(a_2-a_3)(a_2-a_4)\ldots\ldots(a_{n-1}-a_n)\},$$

and it is easy to see that this expression would vanish if any two of the quantities a_1, a_2, a_3, &c. were equal; for, if we suppose that $a_1-a_2=0$, all the terms would vanish except the first two on account of the factor a_1-a_2, and these two would be equal and opposite.

Hence a_1-a_2, a_1-a_3, &c. are all factors of

$$\Sigma a_1^r\{(a_2-a_3)(a_2-a_4)\ldots\ldots(a_{n-1}-a_n)\},$$

and there are $(n-1)+(n-2)+\ldots1=\frac{1}{2}n(n-1)$ of the quantities a_k-a_θ.

But the number of dimensions in the expression

$$\Sigma a_1^r\{(a_2-a_3)(a_2-a_4)\ldots\ldots(a_{n-1}-a_n)\}$$

is $$r+\frac{1}{2}(n-1)(n-2)=\frac{1}{2}n(n-1)+r-n+1.$$

Hence, if $r<n-1$, the expression vanishes for more values than the number of its dimensions: it must therefore [Art. 90] vanish identically.

If $r=n-1$, we have found all the algebraical factors, and we have

$$\Sigma a_1^{n-1}\{(a_2-a_3)\ldots\ldots(a_{n-1}-a_n)\}=L\Pi(a_1-a_2),$$

where L is numerical, and by comparing the coefficients of a_1^{n-1} we see at once that $L=1$.

Lastly, if $r=n$, there will be another algebraical factor, and this must be Σa_1, since $\Sigma a_1{}^r\{(a_2-a_3)\ldots\ldots(a_{n-1}-a_n)\}$ is a cyclically symmetrical expression. Hence

$$\Sigma a_1{}^n\{(a_2-a_3)\ldots\ldots(a_{n-1}-a_n)\}=L(a_1+a_2+\ldots a_n)\cdot\Pi(a_1-a_2),$$

where L is numerical and is found as before to be unity.

54. It is obvious that

$$1+\frac{a_1}{x-a_1}=\frac{x}{x-a_1};$$

hence assume that

$$1+\frac{a_1}{x-a_1}+\ldots+\frac{a_{n-1}x^{n-2}}{(x-a_1)\ldots(x-a_{n-1})}=\frac{x^{n-1}}{(x-a_1)\ldots(x-a_{n-1})}.$$

Then

$$1+\frac{a_1}{x-a_1}+\ldots\frac{a_{n-1}x^{n-2}}{(x-a_1)\ldots(x-a_{n-1})}+\frac{a_nx^{n-1}}{(x-a_1)\ldots(x-a_n)}$$
$$=\frac{x^{n-1}}{(x-a_1)\ldots(x-a_{n-1})}+\frac{a_nx^{n-1}}{(x-a_1)\ldots(x-a_n)}=\frac{x^n}{(x-a_1)\ldots(x-a_n)}.$$

Thus the theorem will be true for $n+1$ terms if it be true for n terms; but the theorem is obviously true for 2 terms, and it must therefore be true for 3; and, being true for 3 terms, it must be true for 4; and so on indefinitely.

55. We have

$$\frac{b+c+d+\ldots+k+l}{a(a+b+c+\ldots+k+l)}-\frac{l}{(a+b+\ldots+k)(a+b+\ldots+k+l)}$$
$$=\frac{(a+b+\ldots+k)(b+c+\ldots+k+l)-al}{a(a+b+\ldots+k)(a+b+\ldots+k+l)}$$
$$=\frac{b+c+\ldots+k}{a(a+b+\ldots+k)}.$$

Similarly

$$\frac{b+c+\ldots+h+k}{a(a+b+\ldots+h+k)}-\frac{k}{(a+\ldots+h)(a+b+\ldots+k)}=\frac{b+c+\ldots+h}{a(a+b+\ldots+h)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{b+c+d}{a(a+b+c+d)}-\frac{d}{(a+b+c)(a+b+c+d)}=\frac{b+c}{a(a+b+c)},$$
$$\frac{b+c}{a(a+b+c)}-\frac{c}{(a+b)(a+b+c)}=\frac{b}{a(a+b)},$$
$$\frac{b}{a(a+b)}-\frac{b}{a(a+b)}=0.$$

The required result follows at once by addition.

X. Pages 130—134.

1. Since

$$(a+b)^3=\{(x-a+2b)-(x-2a+b)\}^3=(x-a+2b)^3-(x-2a+b)^3$$
$$-3(x-a+2b)(x-2a+b)\{(x-a+2b)-(x-2a+b)\},$$

we have $(x-a+2b)(x-2a+b)=0,$

whence $x=a-2b$ or $x=2a-b.$

2. We have

$$(x-1)\{(c+a-2b)x-(b+c-2a)\}=0;$$
$$\therefore\ x=1 \text{ or } x=(b+c-2a)/(c+a-2b).$$

3. $$a^2(x+b)^2-b^2(x-a)^2=0,$$

that is $$\{a(x+b)+b(x-a)\}\{a(x+b)-b(x-a)\}=0,$$

whence $x=0$ or $x=2ab/(b-a).$

4. Multiply both sides by $2(a+x)(b+x)$; then after transposition we have

$$2(a+x)^2+2(b+x)^2-5(a+x)(b+x)=0;$$
$$\therefore\ x^2+(a+b)x-2a^2-2b^2+5ab=0;$$

that is $$(x+2a-b)(x+2b-a)=0;$$

whence $x=b-2a$ or $x=a-2b.$

5. Multiply both sides by $(a+bx)(c+dx)$; then

$$(ax+b)(c+dx)=(a+bx)(cx+d);$$
$$\therefore\ (ad-bc)x^2+bc-ad=0;\ \text{whence } x=\pm 1.$$

6. Multiply up, then after reduction

$$(x^2-1)(ab-1)=0, \text{ whence } x=\pm 1.$$

7. We have

$$\frac{3x-4+7}{x+1}=x^2+2x, \text{ or } x^2+2x=3, \text{ whence } x=1 \text{ or } -3.$$

8. We have

$$x+1+\frac{x^2-5x+4}{x^2-1}-\frac{x^2}{x+1}=0,$$

$$\text{or } x+1+\frac{x-4}{x+1}-\frac{x^2}{x+1}=0, \text{ or } x+1-\frac{x^2-x+4}{x+1}=0;$$

whence $(x+1)^2-x^2+x-4=0,$ or $3x-3=0.$

9. Taking the first and last terms together and also the two middle terms, we have

$$\frac{2x}{x^2-64}+\frac{2x}{x^2-36}=0.$$

Hence $x=0$, or $x^2-36+x^2-64=0$ giving $x=\pm\sqrt{50}$.

10. We have

$$\frac{2}{x+8}-\frac{3}{x+15}+\frac{5}{x+9}-\frac{4}{x+6}=0;$$

$$\therefore\ \frac{6-x}{(x+8)(x+15)}+\frac{x-6}{(x+9)(x+6)}=0;$$

$$\therefore\ x-6=0,\ \text{or}\ (x+9)(x+6)-(x+8)(x+15)=0\ \text{giving}\ x=-\frac{33}{4}.$$

11. $$\frac{2}{2x-3}-\frac{3}{3x+2}-\frac{3}{3x+2}+\frac{1}{x-2}=0;$$

$$\therefore\ \frac{13}{(2x-3)(3x+2)}+\frac{8}{(x-2)(3x+2)}=0;$$

$$\therefore\ 13(x-2)+8(2x-3)=0,\ \&c.$$

12. $$\frac{x-a}{x-b}-1+\frac{x-b}{x-c}-1+\frac{x-c}{x-a}-1=0;$$

whence

$$(b-a)(x-c)(x-a)+(c-b)(x-a)(x-b)+(a-c)(x-b)(x-c)=0;$$

$$\therefore\ x=(a^2c+ab^2+bc^2-3abc)/(a^2+b^2+c^2-bc-ca-ab).$$

13. $$\frac{x+a}{a-x}-1+\frac{x+b}{b-x}-1+\frac{x+c}{c-x}-1=0;$$

$$\therefore\ \frac{x}{a-x}+\frac{x}{b-x}+\frac{x}{c-x}=0;$$

$$\therefore\ x=0\ \text{or}\ (b-x)(c-x)+(c-x)(a-x)+(a-x)(b-x)=0;$$

$$\therefore\ x=0\ \text{or}\ \frac{1}{3}\{a+b+c\pm\sqrt{(a^2+b^2+c^2-bc-ca-ab)}\}.$$

14. $$\frac{x+a}{x-a}-1+\frac{x+b}{x-b}-1+\frac{x+c}{x-c}-1=0;$$

$$\therefore\ a(x-b)(x-c)+b(x-c)(x-a)+c(x-a)(x-b)=0,$$

whence

$$x=[bc+ca+ab\pm\sqrt{\{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)\}}]/(a+b+c).$$

15. $$2-\frac{3}{x+1}+3-\frac{7}{x+2}=4+1-\frac{6}{x-1};$$

$$\therefore\ \frac{3}{x+1}+\frac{7}{x+2}=\frac{6}{x-1};$$

$$\therefore\ 4x^2-15x-25=0, \text{ whence } x=5 \text{ or } -\frac{5}{4}.$$

16. Multiply by $6x$; then $3x^2+12=2x^2+18$, whence $x=\pm\sqrt{6}$.

17. $$\frac{2x^2+2a^2}{x^2-a^2}+\frac{2x^2+2b^2}{x^2-b^2}=0;$$

$$\therefore\ x^4-a^2b^2=0, \text{ \&c.}$$

18. $$\frac{x-1}{x+1}-\frac{x-3}{x+3}=\frac{x-2}{x+2}-\frac{x-4}{x+4};$$

$$\therefore\ \frac{4x}{x^2+4x+3}=\frac{4x}{x^2+6x+8};$$

$$\therefore\ x=0, \text{ or } x^2+4x+3=x^2+6x+8 \text{ giving } x=-\frac{5}{2}.$$

19. $$x+a+\frac{1}{x+b}=x-a+\frac{1}{x-b};$$

$$\therefore\ 2a(x^2-b^2)+x-b=x+b;$$

$$\therefore\ x^2=b^2+\frac{b}{a}.$$

20. Multiply by $(3a-x)(3b-x)(3c-x)$; then

$$(3b-x)(3c-x)+(3c-x)(3a-x)+(3a-x)(3b-x)=0,$$

$$\therefore\ x^2-2(a+b+c)x+3(bc+ca+ab)=0,$$

whence $$x=a+b+c\pm\sqrt{(a^2+b^2+c^2-bc-ca-ab)}.$$

21. $$\frac{a+b}{x+b}-1+\frac{a+c}{x+c}-1=\frac{2(a+b+c)}{x+b+c}-2;$$

$$\therefore\ \frac{a-x}{x+b}+\frac{a-x}{x+c}=\frac{2(a-x)}{x+b+c}.$$

Hence $$x=a, \text{ or } \frac{1}{x+b}+\frac{1}{x+c}=\frac{2}{x+b+c},$$

the latter giving $$(b+c)x=-(b^2+c^2).$$

22. We have

$$\frac{a+c}{x+2b}-\frac{a+c}{x+a+b}+\frac{b+c}{x+2a}-\frac{b+c}{x+a+b}=0;$$

$$\therefore\ \frac{(a+c)(a-b)}{(x+2b)(x+a+b)}-\frac{(b+c)(a-b)}{(x+2a)(x+a+b)}=0;$$

$$\therefore\ (a+c)(x+2a)-(b+c)(x+2b)=0,$$

whence $$x=-2(a+b+c).$$

23. $$\frac{2(a-b)x-a^2+b^2}{(x-a)(x-b)}=\frac{2(a-b)}{x-a-b}.$$

Divide by $a-b$; then

$$\frac{2x-a-b}{(x-a)(x-b)}=\frac{2}{x-a-b};$$

$$\therefore\ (2x-a-b)(x-a-b)=2(x-a)(x-b),$$

whence $$(a+b)x=a^2+b^2.$$

24. We have $$x+\frac{ab}{x+a+b}=x+\frac{cd}{x+c+d};$$

$$\therefore\ ab(x+c+d)=cd(x+a+b),$$

whence $$x=\{cd\ a+b)-ab(c+d)\}/(ab-cd).$$

25. $$(c-d)\left(\frac{a}{x+a}-\frac{b}{x+b}\right)=(a-b)\left(\frac{c}{x+c}-\frac{d}{x+d}\right);$$

$$\therefore\ \frac{(c-d)(a-b)x}{(x+a)(x+b)}=\frac{(a-b)(c-d)x}{(x+c)(x+d)};$$

$$\therefore\ x=0 \text{ or } (x+a)(x+b)=(x+c)(x+d).$$

26. $$\frac{x-a}{b}-\frac{a}{x-b}+\frac{x-b}{a}-\frac{b}{x-a}=0;$$

$$\therefore\ \frac{x^2-(a+b)x}{b(x-b)}+\frac{x^2-(a+b)x}{a(x-a)}=0;$$

$$\therefore\ x^2-(a+b)x=0 \text{ giving } x=0 \text{ or } x=a+b,$$

or $$b(x-b)+a(x-a)=0 \text{ giving } x=(a^2+b^2)/(a+b).$$

27. Multiply by the product of the denominators; then we have

$$-\{(b-c)^2+(c-a)^2+(a-b)^2\}x+3(b-c)(c-a)(a-b)=0.$$

28. $$\frac{x}{(1+2x)(2+3x)}+\frac{x}{(3+4x)(4+5x)}=0,$$

$$\therefore\ x=0,\ \text{or}\ (1+2x)(2+3x)+(3+4x)(4+5x)=0,$$

that is $$26x^2+38x+14=0,$$

giving $$x=\frac{1}{26}(-19\pm\sqrt{-3}).$$

29. We have $$\frac{(x+a)(x+b)}{(x-a)(x-b)}-1=\frac{(x+ma)(x+mb)}{(x-ma)(x-mb)}-1;$$

$$\therefore\ \frac{2(a+b)x}{(x-a)(x-b)}=\frac{2m(a+b)x}{(x-ma)(x-mb)};$$

$$\therefore\ x=0,\ \text{or}\ (x-ma)(x-mb)=m(x-a)(x-b),$$

giving $x^2=mab$.

30. $$\sqrt{(2x+9)}=\sqrt{(x-4)}+\sqrt{(x+1)}.$$

Squaring, we have

$$2x+9=x-4+x+1+2\sqrt{(x-4)}\sqrt{(x+1)},$$

or $$6=\sqrt{(x-4)}\sqrt{(x+1)},$$

and therefore $36=x^2-3x-4$, giving $x=8$ or $x=-5$.

31. $$\sqrt{(x-1)(x-2)}=\sqrt{2}-\sqrt{(x-3)(x-4)}.$$

Squaring, we have

$$(x-1)(x-2)=2+(x-3)(x-4)-2\sqrt{2}\sqrt{(x-3)(x-4)},$$

whence $(2x-6)^2=2(x-3)(x-4)$, giving $x=3$ or $x=2$.

32. $$7x-5+4x-1+2\sqrt{(7x-5)}\sqrt{(4x-1)}$$
$$=7x-4+4x-2+2\sqrt{(7x-4)}\sqrt{(4x-2)};$$

$$\therefore\ (7x-5)(4x-1)=(7x-4)(4x-2),\ \text{whence}\ x=1.$$

33. Square both sides; then

$$a^2-x+b^2+x+2\sqrt{(a^2-x)}\sqrt{(b^2+x)}=a^2+b^2+2ab;$$

$$\therefore\ a^2b^2=(a^2-x)(b^2+x),\ \text{whence}\ x=0\ \text{or}\ x=a^2-b^2.$$

34. Square both sides; then

$$a-x+b-x+2\sqrt{(a-x)(b-x)}=a+b-2x.$$

Hence $$(a-x)(b-x)=0.$$

35. Square both sides; then

$$a-bx+c-dx+2\sqrt{(a-bx)}\sqrt{(c-dx)}=a+c-(b+d)x.$$

Hence $$(a-bx)(c-dx)=0.$$

36. We have $\sqrt{(ax+b^2)} = a - b - \sqrt{(bx+a^2)}$;

$$\therefore\ ax+b^2 = (a-b)^2 + bx + a^2 - 2(a-b)\sqrt{(bx+a^2)};$$

$$\therefore\ x - 2a = -2\sqrt{(bx+a^2)};$$

$$\therefore\ x^2 - 4ax + 4a^2 = 4bx + 4a^2, \text{ whence } x=0 \text{ or } x = 4(a+b).$$

37. Square both sides; then

$$a+x+b+x+2\sqrt{(a+x)}\sqrt{(b+x)} = a+b+2x;$$

$$\therefore\ (a+x)(b+x) = 0.$$

38. Square both sides; then

$$a-x+b+x+2\sqrt{(a-x)}\sqrt{(b+x)} = 2a+2b;$$

$$\therefore\ 2\sqrt{(a-x)}\sqrt{(b+x)} = a+b;$$

$$\therefore\ 4x^2 - 4(a-b)x + (a-b)^2 = 0; \quad \therefore\ x = \frac{1}{2}(a-b).$$

39. Square both sides; then

$$(a+x)(x+b) + (a-x)(x-b) + 2\sqrt{(a^2-x^2)}\sqrt{(x^2-b^2)} = 4ax;$$

$$\therefore\ \sqrt{(a^2-x^2)}\sqrt{(x^2-b^2)} = x(a-b);$$

$$\therefore\ x^4 - 2abx^2 + a^2b^2 = 0; \quad \therefore\ x = \pm\sqrt{ab}.$$

40. Transposing and squaring, we have

$$a(a+b+x) = a(a+b-x) + x^2 + 2x\sqrt{a(a+b-x)},$$

or

$$2ax - x^2 = 2x\sqrt{a(a+b-x)}.$$

Hence $x=0$, or $(2a-x)^2 = 4a(a+b-x)$, giving $x^2 = 4ab$.

41. Transposing and squaring, we have

$$x^2 + ax + b^2 = x^2 - ax + b^2 + 4a^2 + 4a\sqrt{(x^2 - ax + b^2)};$$

$$\therefore\ (x-2a)^2 = 4(x^2 - ax + b^2), \text{ or } 3x^2 = 4(a^2 - b^2).$$

42. Square both sides; then

$$2x^2 + 2a^2 + 2\sqrt{(x^4 + a^2x^2 + a^4)} = 2a^2 - 2b^2;$$

$$\therefore\ (x^2+b^2)^2 = x^4 + a^2x^2 + a^4;$$

$$\therefore\ x^2 = (a^4 - b^4)/(2b^2 - a^2).$$

43. Square both sides; then

$$(a+c)x + 2\sqrt{(ax-b)}\sqrt{(cx+b)} = (a+c)x + 2\sqrt{(ax+b)}\sqrt{(cx-b)};$$

$$\therefore\ (ax-b)(cx+b) = (ax+b)(cx-b), \text{ whence } x=0.$$

44. $$\sqrt{a(b+x-a)} + \sqrt{b(a+x-b)} = -\sqrt{x(a+b-x)};$$

$$\therefore\ a(b+x-a) + b(a+x-b) + 2\sqrt{ab\{x^2 - (a-b)^2\}} = x(a+b-x);$$

$$\therefore\ x^2 - (a-b)^2 = -2\sqrt{ab\{x^2 - (a-b)^2\}},$$

whence

$$x = \pm(a-b) \text{ or } x = \pm(a+b).$$

45. Rationalizing, we have

$$3x^2+2(a+b+c)x-(a^2+b^2+c^2-2bc-2ca-2ab)=0;$$

$$\therefore\ x=\frac{1}{3}\{-(a+b+c)\pm 2\sqrt{(a^2+b^2+c^2-bc-ca-ab)}\}.$$

46. Rationalizing, we have

$$x^2(a^2+ab+b^2)^2+2ab(a+b)(a^2+ab+b^2)x-3a^2b^2(a+b)^2=0,$$

whence $x=ab(a+b)/(a^2+ab+b^2)$ or $x=-3ab(a+b)/(a^2+ab+b^2)$.

47. We have

$$\sqrt{x^2-c^2-a^2}+\sqrt{x^2-a^2-b^2}=x-\sqrt{x^2-b^2-c^2};$$

$$\therefore\ a^2-x\sqrt{(x^2-b^2-c^2)}=\sqrt{(x^2-c^2-a^2)}\sqrt{(x^2-a^2-b^2)};$$

$$\therefore\ a^4+x^2(x^2-b^2-c^2)-2a^2x\sqrt{(x^2-b^2-c^2)}=(x^2-c^2-a^2)(x^2-a^2-b^2),$$

or $$2a^2x\sqrt{(x^2-b^2-c^2)}=2a^2x^2-b^2c^2-c^2a^2-a^2b^2,$$

whence $$4a^2b^2c^2x^2=(b^2c^2+c^2a^2+a^2b^2)^2.$$

48. We have

$$\sqrt{b^2-x^2}+\sqrt{c^2-x^2}=\sqrt{a^2+b^2+c^2-x^2}-\sqrt{a^2-x^2}.$$

Squaring and transposing

$$\sqrt{(b^2-x^2)}\sqrt{(c^2-x^2)}-a^2=-\sqrt{(a^2-x^2)}\sqrt{(a^2+b^2+c^2-x^2)};$$

$$\therefore\ 2a^2\sqrt{(b^2-x^2)}\sqrt{(c^2-x^2)}=b^2c^2-c^2a^2-a^2b^2+2a^2x^2;$$

$$\therefore\ 4a^2b^2c^2x^2=4a^4b^2c^2-(b^2c^2-c^2a^2-a^2b^2)^2;$$

$$\therefore\ 4a^2b^2c^2x^2=(bc+ca+ab)(-bc+ca+ab)(bc-ca+ab)(bc+ca-ab).$$

49. $\sqrt{14-(3x-2)(x-1)}=\sqrt{12+5x-3x^2}=\sqrt{-(x-3)(3x+4)}$.

In order that the expression may be real $(x-3)(3x+4)$ must be negative, and therefore x cannot be positive and greater than 3 nor negative and numerically greater than $\frac{4}{3}$.

50. Put $$\frac{x^2+34x-71}{x^2+2x-7}=\lambda;$$

then $$x^2(1-\lambda)+2x(17-\lambda)+7\lambda-71=0.$$

Now, if x be real, $$(17-\lambda)^2-(1-\lambda)(7\lambda-71)\not<0,$$

or $$8(\lambda-9)(\lambda-5)\not<0;$$

hence λ cannot lie between 5 and 9.

51. Put $$\frac{x^2-6x+5}{x^2+2x+1}=\lambda;$$

$$\therefore\ x^2(1-\lambda)-2x(3+\lambda)+(5-\lambda)=0.$$

Hence, if x be real, we must have

$$(3+\lambda)^2-(1-\lambda)(5-\lambda)\not<0, \text{ or } 12\lambda+4\not<0, \text{ or } \lambda\not<-\frac{1}{3}.$$

52. Put $(x^2-x+1)/(x^2+x+1)=\lambda$; then

$$x^2(1-\lambda)-x(1+\lambda)+(1-\lambda)=0.$$

Hence, as x is real, $(1+\lambda)^2-4(1-\lambda)^2 \leqslant 0$, whence $(3\lambda-1)(\lambda-3) \geqslant 0$, so that λ must lie between $\frac{1}{3}$ and 3.

53. We have $(x-3)^2+(y+4)^2=25$. Hence $(x-3)^2 \geqslant 25$, whence it follows that if x be positive it must not be greater than 8, and if x be negative it must not be numerically greater than 2. Similarly y must lie between -9 and 1.

54. We have $(x-4)^2+(2y-4)^2=6^2$. Hence $(x-4)^2 \geqslant 6^2$, whence it follows that x lies between -2 and 10. Also $(2y-4)^2 \leqslant 6^2$, whence it follows that y lies between -1 and 5.

55. We have $x^4+2x^2(y^2-a^2)+y^4+2a^2y^2=0$. Hence, for real values of x^2, $(y^2-a^2)^2 \leqslant y^4+2a^2y^2$, or $a^2 \leqslant 4y^2$, so that $\frac{a}{2}$ is the greatest possible value of y.

56. We have $(bx+a)^2+(b'x+a')^2=0$. Hence, as the squares of real quantities must be positive, and the sum of two positive quantities cannot be zero unless the quantities are separately zero, we must have $bx+a$ and $b'x+a'$ simultaneously zero. This can only be the case for *one* value of x, and is not true unless $\frac{a}{b}=\frac{a'}{b'}$.

57. Let the roots be $m\lambda$ and $n\lambda$; then $(m+n)\lambda=-\frac{b}{a}$ and $mn\lambda^2=\frac{c}{a}$.

Hence $$(m+n)^2\frac{c}{a}=\frac{b^2}{a^2}mn.$$

58. We have

$$\frac{-b+\sqrt{(b^2-ac)}}{a}=\frac{-b'+\sqrt{(b'^2-a'c')}}{a'};$$

$$\therefore\ (ab'-a'b)-a\sqrt{(b'^2-a'c')}=-a'\sqrt{(b^2-ac)};$$

$$\therefore\ 2ab'^2-a'(2bb'+ac'-a'c)=2(ab'-a'b)\sqrt{(b'^2-a'c')}.$$

Hence $b'^2-a'c'$ must be rational, or else $ab'-a'b=0$, and therefore $\frac{a}{b}=\frac{a}{b'}$.

Now, if $\frac{a}{b}=\frac{a'}{b'}$, the sum of the roots of one equation is equal to the sum of the roots of the other; and hence in this case *each* of the roots of one equation is equal to a root of the other.

Hence, if the equations have only *one* root in common, $b'^2-a'c'$ must be rational, and similarly b^2-ac must be rational.

59. We have $x_1+x_2=-b/a$ and $x_1x_2=c/a$.

(i) The required equation is $(x-x_1^3)(x-x_2^3)=0$.

Now $x_1^3+x_2^3=(x_1+x_2)\{(x_1+x_2)^2-3x_1x_2\}=-\frac{b}{a}\left\{\frac{b^2}{a^2}-\frac{3c}{a}\right\}$,

and $x_1^3x_2^3=-\frac{c^3}{a^3}$.

Hence the required equation is

$$a^3x^2+b(b^2-3ac)x+c^3=0.$$

(ii) The required equation is $\left(x-\frac{x_1^2}{x_2}\right)\left(x-\frac{x_2^2}{x_1}\right)=0$,

or $$x^2-\frac{x_1^3+x_2^3}{x_1x_2}x+x_1x_2=0,$$

or $$a^2cx^2+b(b^2-3ac)x+ac^2=0.$$

(iii) The required equation is $\{x-(b+ax_1)\}\{x-(b+ax_2)\}=0$,

or $$(x-b)^2-a(x-b)(x_1+x_2)+a^2x_1x_2=0,$$

or $$(x-b)^2+b(x-b)+ac=0.$$

60. (i) $bx_1x_2(x_1+x_2)+c(x_1^2+x_2^2)$

$$=b\cdot\frac{c}{a}\left(-\frac{b}{a}\right)+c\left(\frac{b^2}{a^2}-2\frac{c}{a}\right)=-2\frac{c^2}{a}.$$

(ii) $2b^2x_1^2x_2^2+2bcx_1x_2(x_1+x_2)+c^2\{(x_1+x_2)^2-2x_1x_2\}$

$$=[2b^2c^2-2b^2c^2+c^2(b^2-2ac)]\div a^2=c^2(b^2-2ac)\div a^2.$$

61. $x_1^2+x_1x_2+x_2^2+a=(x_1+x_2)^2-x_1x_2+a=(-m)^2-m^2-a+a=0.$

62. We have (i) $x_1+x_2=\frac{-ma}{1+(1-m)a^2}$ and (ii) $x_1x_2=\frac{1+a^2}{1+(1-m)a^2}$.

Hence

$$(x_1^2+1)(x_2^2+1)=(x_1+x_2)^2+(1-x_1x_2)^2=(m^2a^2+m^2a^4)/\{1+(1-m)a^2\}^2$$

$$=\frac{m(1+a^2)}{1+(1-m)a^2}\times\frac{ma^2}{1+(1-m)a^2}=mx_1x_2(x_1x_2-1).$$

63. We have $x_1+x_2=-Am/(A+Bm^2)$, and $x_1x_2=Am^2/(A+Bm^2)$;

$$\therefore\ x_1^2+x_2^2=\{A^2m^2-2Am^2(A+Bm^2)\}/(A+Bm^2)^2$$

$$=-(A^2m^2+2ABm^4)/(A+Bm^2)^2.$$

Hence $$A(x_1^2+x_2^2)+Ax_1x_2+Bx_1^2x_2^2$$

$$=\{-A^3m^2-2A^2Bm^4+A^2m^2(A+Bm^2)+BA^2m^4\}/(A+Bm^2)^2=0.$$

64. Put $2(a-x)(x+\sqrt{x^2+b^2})=u$; then

$\{u-2x(a-x)\}^2=4(a-x)^2(x^2+b^2)$; $\therefore\ u^2-4ux(a-x)-4b^2(a-x)^2=0$,

or $$u^2-4a^2b^2-4ax(u-2b^2)+4x^2(u-b^2)=0.$$

Hence, as x is real, we have $a^2(u-2b^2)^2>(u-b^2)(u^2-4a^2b^2)$,

whence $$u^2(a^2+b^2-u)>0.$$

65. Put λ for the given expression; then we have

$$(2-\lambda)x^4-4x^3+(9-2\lambda)x^2-4x+2-\lambda=0,$$

or
$$(2-\lambda)\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+9-2\lambda=0,$$

or
$$(2-\lambda)\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)+5=0.$$

Now, since x is real, $x+\frac{1}{x}$ is also real, and therefore

$$4-5(2-\lambda)\not<0, \text{ or } \lambda\not<\frac{6}{5}.$$

XI. Pages 141—143.

1. We have $(x^2-4)(x^2+2)=0,$

whence $x=\pm 2$ or $x=\pm\sqrt{-2}.$

2. We have $(x^3+8a^3)(x^3-a^3)=0,$

whence $x^3+8a^3=0,$ or $x^3-a^3=0.$

Hence the roots are $-2a,\ -2a\omega,\ -2a\omega^2,\ a,\ a\omega,\ a\omega^2$, where $1,\ \omega,\ \omega^2$ are the cube roots of unity.

3. We have $(x^3-8a^3)(x^3+a^3)=0,$

whence the roots are $2a,\ 2a\omega,\ 2a\omega^2,\ -a,\ -a\omega,\ -a\omega^2.$

4. Multiply by $2x(x^2+1)$; then

$$2x^2-5x(x^2+1)+2(x^2+1)^2=0,$$

that is $$\{2x-(x^2+1)\}\{x-2(x^2+1)\}=0.$$

Hence $$x=1 \text{ or } x=\frac{1}{4}(1\pm\sqrt{-15}).$$

5. Put $(x^2+2)/(x^2+4x+1)=y$; then $2y^2-5y+2=0,$

whence $$y=\frac{1}{2} \text{ or } y=2.$$

Hence $$\frac{x^2+2}{x^2+4x+1}=\frac{1}{2} \text{ giving } x=1 \text{ or } x=3;$$

or $$\frac{x^2+2}{x^2+4x+1}=2 \text{ giving } x=0 \text{ or } x=-8.$$

6. We have $(x^2+x+1)^2+(x^2+x+1)-12=0,$

that is $\{x^2+x+1+4\}\{x^2+x+1-3\}=0;$

i.e. $x^2+x+5=0$ giving $x=\frac{1}{2}(-1\pm\sqrt{-19}),$

or $x^2+x-2=0$ giving $x=1$ or $x=-2.$

7. We have

$$(x^2+7x+5)^2-3(x^2+7x+5)-4=0;$$

$$\therefore \{x^2+7x+5+1\}\{x^2+7x+5-4\}=0.$$

Hence $x^2+7x+6=0$ giving $x=-1$ or $x=-6;$

or $x^2+7x+1=0$ giving $x=\frac{1}{2}(-7\pm\sqrt{45}).$

8. Put $16-7x-x^2=y^2;$

then $y=16-y^2-\frac{1}{4}$ giving $y=\frac{7}{2}$ or $y=-\frac{9}{2}.$

Hence $16-7x-x^2=\frac{49}{4}$ giving $x=\frac{1}{2}$ or $x=-\frac{15}{2};$

or $16-7x-x^2=\frac{81}{4}$ giving $x=\frac{1}{2}(-7\pm4\sqrt{2}).$

9. Put $x^2-2x+6=y^2;$

then $y^2+6y-27=0;$

whence $y=3$ or $y=-9.$

Hence $x^2-2x+6=9$ giving $x=-1$ or $x=3;$

or $x^2-2x+6=81$ giving $x=1\pm\sqrt{76}.$

10. We have

$$(a-1)(1+x+x^2)^2=(a+1)(1+x+x^2)(1-x+x^2);$$

$$\therefore \quad 1+x+x^2=0 \text{ giving } x=\frac{1}{2}(-1\pm\sqrt{-3});$$

or $(a-1)(1+x+x^2)=(a+1)(1-x+x^2)$

giving $x=\frac{1}{2}\{a\pm\sqrt{(a^2-4)}\}.$

11. Take the first and last factors together and the middle factors together; then

$$(x^2+5x+4)(x^2+5x+6)=24; \text{ or } (x^2+5x)^2+10(x^2+5x)=0.$$

Hence $x^2+5x=0$ giving $x=0$ or $x=-5;$

or $x^2+5x+10=0$ giving $x=\frac{1}{2}(-5\pm\sqrt{-15}).$

12. Take the first and last factors together and the middle factors together, then we have

$$(x^2+8ax+7a^2)(x^2+8ax+15a^2)=384a^4;$$

$$\therefore\ (x^2+8ax)^2+22a^2(x^2+8ax)-279a^4=0,$$

that is $$(x^2+8ax+31a^2)(x^2+8ax-9a^2)=0.$$

Hence $x^2+8ax-9a^2=0$ giving $x=a$ or $x=-9a$;

or $x^2+8ax+31a^2=0$ giving $x=a(-4\pm\sqrt{-15})$.

13. Take the first and last factors together and the middle factors together; then

$$(x^2+ax-12a^2)(x^2+ax-2a^2)=2376a^4;$$

$$\therefore\ (x^2+ax)^2-14a^2(x^2+ax)-2352a^4=0;$$

$$\therefore\ (x^2+ax-56a^2)(x^2+ax+42a^2)=0.$$

Hence $x^2+ax-56a^2=0$ giving $x=7a$ or $x=-8a$;

or $x^2+ax+42a^2=0$ giving $x=\frac{a}{2}(-1\pm\sqrt{-167})$.

14. Take the first and last factors together and the middle factors together; then

$$(x^2+14x+24)(x^2+11x+24)=4x^2;$$

$$\therefore\ (x^2+24)^2+25x(x^2+24)+150x^2=0;$$

i.e. $$(x^2+24+15x)(x^2+24+10x)=0.$$

Hence $x^2+10x+24=0$ giving $x=-6$ or $x=-4$;

or $x^2+15x+24=0$ giving $x=\frac{1}{2}(-15\pm\sqrt{129})$.

15. We have $x^2-3x+4-2x\sqrt{(x^2-3x+4)}+x^2=25$;

$$\therefore\ \sqrt{(x^2-3x+4)}-x=\pm5;$$

$$\therefore\ (x\pm5)^2=x^2-3x+4,$$

whence $$x=3 \text{ or } x=-\frac{21}{13}.$$

16. We have $\{x^2-(a+b)\}^2=0$; $\therefore\ x=\pm\sqrt{(a+b)}$.

17. We have $x^4-2x^2(a^2+b^2)+(a^2+b^2)^2-4a^2b^2=0$;

$$\therefore\ \{x^2-(a^2+b^2)-2ab\}\{x^2-(a^2+b^2)+2ab\}=0;$$

$$\therefore\ x=\pm(a+b) \text{ or } x=\pm(a-b).$$

18. We have $4\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)-7=0.$

Put $x+\frac{1}{x}=y$; then $4(y^2-2)-4y-7=0,$

whence $y=\frac{5}{2}$ or $y=-\frac{3}{2}.$

Then $x+\frac{1}{x}=\frac{5}{2}$ gives $x=2$ or $x=\frac{1}{2},$

and $x+\frac{1}{x}=-\frac{3}{2}$ gives $x=\frac{1}{4}(-3\pm\sqrt{-7}).$

19. We have $9\left(x^2+\frac{1}{x^2}\right)-24\left(x+\frac{1}{x}\right)-2=0.$

Put $x+\frac{1}{x}=y$; then $9(y^2-2)-24y-2=0,$

whence $y=\frac{10}{3}$ or $y=-\frac{2}{3}.$

Then $x+\frac{1}{x}=\frac{10}{3}$ gives $x=3$ or $x=\frac{1}{3},$

and $x+\frac{1}{x}=-\frac{2}{3}$ gives $x=\frac{1}{3}\left(-1\pm\sqrt{-8}\right).$

20. We have $(x+1)(x^4-x^3+x^2-x+1)=0.$

Hence $x=-1$ or else $x^4-x^3+x^2-x+1=0$, and therefore

$$\left(x+\frac{1}{x}\right)^2-\left(x+\frac{1}{x}\right)-1=0;$$

$$\therefore\ x+\frac{1}{x}=\frac{1}{2}(1\pm\sqrt{5}),$$

whence $x=\frac{1}{4}[1+\sqrt{5}\pm\sqrt{(-10+2\sqrt{5})}]$ or $x=\frac{1}{4}[1-\sqrt{5}\pm\sqrt{(-10-2\sqrt{5})}].$

21. We have $(x^4+1)(x^2+1)(x^2-1)=0.$

Hence $x=\pm1$; or $x=\pm\sqrt{-1}$; or $x^4+1=0,$

that is $(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)=0,$

whence $x=\frac{1}{2}\{\pm\sqrt{2}\pm\sqrt{-2}\}.$

22. We find by trial that $x=2$ satisfies the equation; hence $x-2$ is a factor of the expression on the left, and we have

$$(x-2)(3x^2-8x+4)=0,$$

that is

$$(x-2)^2(3x-2)=0.$$

Hence the roots are 2, 2, $\frac{2}{3}$.

23. We find by trial that $x=-1$ satisfies the equation. Hence we have

$$(x+1)(x^3-x^2-14x+24)=0.$$

Again $x=3$ satisfies $x^3-x^2-14x+24=0$; hence we have

$$(x+1)(x-3)(x^2+2x-8)=0.$$

Hence the roots are -1, 3, 2 and -4.

24. We have $x^4-1+7x(x^2-1)=0$, that is $(x^2-1)(x^2+1+7x)=0$.

Hence

$$x=\pm 1 \text{ or } x=\frac{1}{2}(-7\pm\sqrt{45}).$$

25. The equation is satisfied by the values $x=a$ or $x=b$ or $x=c$; and as the equation is of the *third* degree there are no other roots.

26. The equation is obviously satisfied by the value $x=9$, and the equation may be written $(x-9)(x^2+6x+56)=0$.

Hence

$$x=9 \text{ or } x=-3\pm\sqrt{-47}.$$

27. The equation is obviously satisfied by the values $x=9$ or $x=-6$: hence $(x-9)(x+6)$ is a factor of $x(x-1)(x-2)(x-3)-9$, 8, 7, 6, and we have

$$(x-9)(x+6)(x^2-3x+56)=0.$$

Hence the roots are 9, -6, $\frac{1}{2}(3\pm\sqrt{-215})$.

28. We have $(a-x)^3+(b-x)^3=\{(a-x)+(b-x)\}^3$;

$$\therefore 3(a-x)(b-x)(a+b-2x)=0, \text{ \&c.}$$

29. We have $(a-x)^4+(b-x)^4=\{(a-x)+(b-x)\}^4$;

$$\therefore (a-x)(b-x)\{4(a-x)^2+6(a-x)(b-x)+4(b-x)^2\}=0.$$

Hence the roots are a, b and the roots of the quadratic

$$7x^2-7x(a+b)+2a^2+3ab+2b^2=0.$$

30. $(a-x)^5+(b-x)^5=(a-x+b-x)^5=(a-x)^5+5(a-x)^4(b-x)$
$+10(a-x)^3(b-x)^2+10(a-x)^2(b-x)^3+5(a-x)(b-x)^4+(b-x)^5$.

Hence $(a-x)(b-x)[(a-x)^3+(b-x)^3+2(a-x)(b-x)(a+b-2x)]=0,$

or $(a-x)(b-x)(a+b-2x)[(a-x)^2+(a-x)(b-x)+(b-x)^2]=0.$

Hence the roots are a, b, $\frac{1}{2}(a+b)$ and the roots of the quadratic

$$3x^2-3(a+b)x+a^2+ab+b^2=0.$$

31. Cubing both sides, we have

$$a-x+b-x+3\sqrt[3]{(a-x)}\sqrt[3]{(b-x)}\{\sqrt[3]{(a-x)}+\sqrt[3]{(b-x)}\}=a+b-2x;$$

$$\therefore\ x=a \text{ or } x=b \text{ or } \sqrt[3]{(a-x)}+\sqrt[3]{(b-x)}=0.$$

But, if $\sqrt[3]{(a-x)}=-\sqrt[3]{(b-x)}$, we have by cubing both sides $a-x=-(b-x)$, so that $x=\frac{1}{2}(a+b)$.

32. $\sqrt[4]{(a-x)}+\sqrt[4]{(b-x)}=\sqrt[4]{(a+b-2x)}.$

Hence $(a-x)+2\sqrt[4]{(a-x)}\sqrt[4]{(b-x)}[2\sqrt[2]{(a-x)}+3\sqrt[4]{(a-x)}\sqrt[4]{(b-x)}+2\sqrt[2]{(b-x)}]$
$+b-x=a+b-2x;$

$$\therefore\ \sqrt[4]{(a-x)}\cdot\sqrt[4]{(b-x)}[2\sqrt[2]{(a-x)}+3\sqrt[4]{(a-x)}\sqrt[4]{(b-x)}+2\sqrt[2]{(b-x)}]=0.$$

Hence $x=a$ or $x=b$

or $2\sqrt[2]{(a-x)}+2\sqrt[2]{(b-x)}=-3\sqrt[4]{(a-x)}\sqrt[4]{(b-x)}.$

Squaring both sides of the last equation, we have after transposition

$$4(a+b-2x)=\sqrt[2]{(a-x)}\cdot\sqrt[2]{(b-x)};$$

$$\therefore\ 16(a+b-2x)^2=(a-x)(b-x);$$

or $$63x^2-63(a+b)x+16(a+b)^2-ab=0,$$

whence $$x=\frac{1}{2}\left\{(a+b)\pm\frac{1}{63}(a-b)\sqrt{-63}\right\}.$$

Thus the required roots are a, b,

$$\frac{1}{2}\left\{(a+b)\pm\frac{1}{63}(a-b)\sqrt{-63}\right\}.$$

33. We have

$$(a-x)^5+(x-b)^5=(a-x+x-b)^5=(a-x)^5+5(a-x)^4(x-b)+10(a-x)^3(x-b)^2$$
$$+10(a-x)^2(x-b)^3+5(a-x)(x-b)^4+(x-b)^5;$$

$$\therefore\ 5(a-x)(x-b)\{(a-x)^3+(x-b)^3+2(a-x)(x-b)(a-b)\}=0.$$

Hence $x=a$ or $x=b$ or $(a-x)^2+(a-x)(x-b)+(x-b)^2=0,$

the last equation giving

$$x=\frac{1}{2}\left\{a+b\pm(a-b)\sqrt{-3}\right\}.$$

34. Cubing both sides, we have

$$a-x+x-b+3\sqrt[3]{(a-x)}\sqrt[3]{(x-b)}\{\sqrt[3]{(a-x)}+\sqrt[3]{(x-b)}\}=a-b;$$

$$\therefore\ x=a \text{ or } x=b \text{ or } \sqrt[3]{(a-x)}+\sqrt[3]{(x-b)}=0.$$

If $\sqrt[3]{(a-x)}=-\sqrt[3]{(x-b)}$, we have $a-x=-(x-b)$ which requires that x should be infinite unless $a-b=0$.

35. Raising both sides to the fourth power and cancelling we have

$$\sqrt[4]{(a-x)}\sqrt[4]{(x-b)}\,[2\sqrt[2]{(a-x)}+3\sqrt[4]{(a-x)}\sqrt[4]{(x-b)}+2\sqrt[2]{(x-b)}]=0.$$

Hence $x=a$ or $x=b$ or $2\sqrt[2]{(a-x)}+2\sqrt[2]{(x-b)}=-3\sqrt[4]{(a-x)}\sqrt[4]{(x-b)}$.

From the last equation, by squaring and transposing, we have

$$4(a-b)=\sqrt{(a-x)}\sqrt{(x-b)}, \text{ or } x^2-x(a+b)+16(a-b)^2+ab=0,$$

giving $$x=\frac{1}{2}\left\{(a+b)\pm(a-b)\sqrt{-63}\right\}.$$

36. Put $x=y+\frac{1}{2}a$; then

$$(2y+a)^4+(a-2y)^4=16b^4, \text{ or } 16y^4+24y^2a^2+a^4=8b^4,$$

whence the result.

37. Put $$x=y-\frac{1}{2}(a+b);$$

then $$\left\{y+\frac{1}{2}(a-b)\right\}^4+\left\{y-\frac{1}{2}(a-b)\right\}^4=17(a-b)^4;$$

$$\therefore\ 2y^4+3y^2(a-b)^2+\frac{1}{8}(a-b)^4=17(a-b)^4,$$

or $$16y^4+24(a-b)^2y^2-135(a-b)^4=0;$$

$$\therefore\ \{4y^2-9(a-b)^2\}\{4y^2+15(a-b)^2\}=0.$$

Hence $$y=\pm\frac{3}{2}(a-b),$$

and therefore $$x=a-2b \text{ or } x=b-2a;$$

or $$y=\frac{1}{2}(a-b)\sqrt{(-15)},$$

and therefore $$x=\frac{1}{2}\{-(a+b)\pm(a-b)\sqrt{-15}\}.$$

38. We have by raising to the fourth power

$$x+(a-x)+2\sqrt[4]{x}\sqrt[4]{a-x}\,[2\sqrt[2]{x}+3\sqrt[4]{x}\sqrt[4]{(a-x)}+2\sqrt[2]{(a-x)}]=b;$$

and by squaring $$\sqrt[2]{x}+2\sqrt[4]{x}\sqrt[4]{(a-x)}+\sqrt[2]{(a-x)}=\sqrt{b},$$

$$\therefore\ 2\sqrt[4]{x}\sqrt[4]{(a-x)}\,[2\sqrt{b}-\sqrt[4]{x}\sqrt[4]{(a-x)}]=b-a.$$

Hence, if y is put for $\sqrt[4]{x}\sqrt[4]{(a-x)}$, we have $2y(2\sqrt{b}-y)=b-a$,

whence $$y=\sqrt{b}\pm\sqrt{\frac{1}{2}(a+b)}.$$

Hence x is given by $x(a-x)=\left\{\sqrt{b}\pm\sqrt{\frac{1}{2}(a+b)}\right\}^4$.

39. The equation may be written

$$(x^2-ab)(a^2-bx)(b^2-ax)=0.$$

Hence the roots are $\frac{a^2}{b}$, $\frac{b^2}{a}$ and $\pm\sqrt{ab}$.

40. We have

$$x^3abc-x^2(b^2c^2+c^2a^2+a^2b^2)+xabc(a^2+b^2+c^2)-a^2b^2c^2=0,$$

that is $$(ax-bc)(bx-ca)(cx-ab)=0,$$

whence $$x=\frac{bc}{a} \text{ or } x=\frac{ca}{b} \text{ or } x=\frac{ab}{c}.$$

41. It is obvious by inspection that $x=0$ and $x=a+b$ satisfy the equation. Hence, if the equation be written in the form

$$(a+b)^2\{(a-x)^4+(x-b)^4\}-(a^4+b^4)(a+b-2x)^2=0,$$

$x(x-a-b)$ will be a factor of the expression on the left, and the other factor is easily found to be

$$2(a+b)^2x^2-2(a+b)^3x+4ab(a^2+b^2)=0,$$

that is $$\{(a+b)x-2ab\}\{(a+b)x-(a^2+b^2)\}=0.$$

Hence the required roots are

$$0,\ a+b,\ 2ab/(a+b) \text{ and } (a^2+b^2)/(a+b).$$

42. We have

$$x^4+b^2x^3-2b^2x^2-b^4x+b^4+abx(x^2+b^2x-b^2)=0;$$

$$\therefore\ (x^2+b^2x-b^2)\{x^2-b^2+abx\}=0.$$

Hence $x^2+b^2x-b^2=0$ giving $x=\frac{b}{2}\{-b\pm\sqrt{(b^2+4)}\}$,

or $x^2+abx-b^2=0$ giving $x=\frac{b}{2}\{-a\pm\sqrt{(a^2+4)}\}$.

43. We have $(x^2+b^2)^2-2ax(x^2+b^2)+a^2x^2=0$;

$$\therefore\ x^2+b^2-ax=0;$$

$$\therefore\ x=\frac{1}{2}\{a\pm\sqrt{(a^2-4b^2)}\}.$$

44. It is obvious that $-a-b-c$ is one root. The equation may therefore be written $(x+a+b+c)\{x^2+(a+b+c)x+bc+ca+ab\}=0$.

45. We have $$\frac{a}{b+c-x}+1+\frac{b}{c+a-x}+1+\frac{c}{a+b-x}+1=0,$$

that is $$(a+b+c-x)\left(\frac{1}{b+c-x}+\frac{1}{c+a-x}+\frac{1}{a+b-x}\right)=0.$$

Hence one root is $a+b+c$, and the others are the roots of

$$3x^2-4x(a+b+c)+\Sigma a^2+3\Sigma bc=0.$$

46. Multiply by the L.C.M. of the denominators, namely

$$(x+a-b-c)(x-a+b-c)(x-a-b+c);$$

then $$\Sigma(x-a)^2(x+a-b-c)=(x+a-b-c)(x-a+b-c)(x-a-b+c);$$

$$\therefore\quad 3x^3-3x^2(a+b+c)+x(-\Sigma a^2+4\Sigma bc)+\Sigma a^3-\Sigma a^2b=x^3-x^2\Sigma a$$
$$+x(2\Sigma bc-\Sigma a^2)+\Sigma a^3-\Sigma a^2b+2abc;$$

$$\therefore\quad x^3-(a+b+c)x^2+x\Sigma bc-abc=0,$$

that is $$(x-a)(x-b)(x-c)=0.$$

47. We have

$$1+\frac{2x(a+b)}{(x-a)(x-b)}+1-\frac{2x(a+b)}{(x+a)(x+b)}=1+\frac{2x(c+d)}{(x-c)(x-d)}+1-\frac{2x(c+d)}{(x+c)(x+d)}.$$

Hence one root is zero, and the others are given by

$$(a+b)\left\{\frac{1}{(x-a)(x-b)}-\frac{1}{(x+a)(x+b)}\right\}$$
$$=(c+d)\left\{\frac{1}{(x-c)(x-d)}-\frac{1}{(x+c)(x+d)}\right\};$$

$$\therefore\quad \frac{(a+b)^2x}{(x^2-a^2)(x^2-b^2)}=\frac{(c+d)^2x}{(x^2-c^2)(x^2-d^2)}.$$

Hence another root is zero, and the others are given by

$$(a+b)^2(x^2-c^2)(x^2-d^2)-(c+d)^2(x^2-a^2)(x^2-b^2)=0,$$

or $$\{(a+b+c+d)x^2+cd(a+b)+ab(c+d)\}\{(a+b-c-d)x^2-ab(c+d)$$
$$+cd(a+b)\}=0.$$

It should be remarked that terms of the eighth degree would be introduced by multiplying throughout by the L.C.M. of the denominators: two of the roots of this equation are zero, four are given by

$$\pm\sqrt{\left\{-\frac{cd(a+b)+ab(c+d)}{a+b+c+d}\right\}}\quad\text{and}\quad\pm\sqrt{\left\{\frac{ab(c+d)-cd(a+b)}{a+b-c-d}\right\}},$$

and the remaining two are infinite.

XII. Pages 155—157.

1. Multiply by 6 and 10 respectively; then $2x-y=3$ and $2x-3y=5$. Hence by subtraction $2y=-2$ or $y=-1$. Then $2x+1=3$; $\therefore\ x=1$.

2. We have $\frac{18}{x}-\frac{8}{y}=4$, $\frac{18}{x}+\frac{8}{y}=10$; $\therefore \frac{36}{x}=14$ and $\frac{16}{y}=6$.

3. Multiply the first by 3 and subtract the second; then

$$\frac{11}{y}=\frac{21}{2}-\frac{26}{3};\ \text{therefore } y=6.\ \text{Then } x+\frac{3}{6}=\frac{7}{2};\ \therefore\ x=3.$$

4.
$$\frac{4}{x}-\frac{3}{y}=5 \text{ and } \frac{6}{x}+\frac{3}{y}=10.$$

By addition $\frac{10}{x}=15$ or $x=\frac{2}{3}$. Then $6-\frac{3}{y}=5$ or $y=3$.

5.
$$\frac{x}{b(b^2-a^2)+2a^2b}=\frac{y}{2ab^2-a(b^2-a^2)}=\frac{-1}{-a^2-b^2};$$
$$\therefore\ x=\frac{b(b^2+a^2)}{a^2+b^2}=b \text{ and } y=\frac{a(a^2+b^2)}{a^2+b^2}=a.$$

6.
$$\frac{x}{ab^2-a^2b}=\frac{y}{a^2-b^2}=\frac{1}{b-a};\ \therefore\ x=ab,\ y=-(a+b).$$

7. We have $x+\frac{a-b}{a+b}x=2a$; $\therefore\ x=a+b$. Hence $y=2a-(a+b)=a-b$.

8.
$$\frac{x}{2ac(b-c)-2ab(c-a)}=\frac{y}{2ab(c+a)-2ac(b+c)}$$
$$=\frac{-1}{(b+c)(c-a)-(b-c)(c+a)};$$
$$\therefore\ x=\frac{2a(c^2-ab)}{2c^2-2ab}=a, \text{ and } y=\frac{2a(c^2-ab)}{2c^2-2ab}=a.$$

9.
$$\frac{x}{a(a^3+b^3)-2ab^3}=\frac{y}{2a^3b-b(a^3+b^3)}=\frac{-1}{b^3-a^3};$$
$$\therefore\ x=\frac{a(a^3-b^3)}{a^3-b^3}=a, \text{ and } y=\frac{b(a^3-b^3)}{a^3-b^3}=b.$$

10.
$$\frac{x}{b(a^3-b^3)-(a+b)(a^3-b^3)}=\frac{y}{a(a^3-b^3)-(a+b)(a^3-b^3)}=\frac{-1}{(a+b)^2-ab};$$
$$\therefore\ x=\frac{a(a^3-b^3)}{a^2+ab+b^2}=a(a-b), \text{ and } y=\frac{b(a^3-b^3)}{a^2+ab+b^2}=b(a-b).$$

11. Subtract the first from the second and also from the third; then $x+2y=3$ and $3x+8y=15$. Multiply the former by 3 and subtract from the latter; then $2y=6$ or $y=3$. Since $y=3$, $x+6=3$, whence $x=-3$; and then $-3+3+z=1$, whence $z=1$.

12. We have $x+y+z=1$, $2x+y+16z=4$ and $20x+9y-6z=12$. Subtracting the first from the second, and also subtracting nine times the first from the third, we have $x+15z=3$ and $11x-15z=3$.

Hence $12x=6$, whence $x=\frac{1}{2}$. Then $15z=3-\frac{1}{2}$, whence $z=\frac{1}{6}$. Also

$$y=1-\frac{1}{2}-\frac{1}{6}=\frac{1}{3}.$$

13. We have $2x-y-z=0$, $x-2y+z=0$ and $2x+3y+z=6$. Add the first and second, and also the first and third; then $x-y=0$ and $4x+2y=6$. Hence $6x=6$, whence $x=1=y$ and then $z=6-2-3=1$.

14. Subtract the first equation from the sum of the other two; then $2x=2b+2c-2a$. Hence we have $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$.

15. Add the second and third; then $x=b+c$. Similarly $y=c+a$, and $z=a+b$.

16. Multiply the first by 2 and add the second and third; then

$$-4x=4a+2b+2c.$$

Hence we have

$$x=-\frac{1}{2}(2a+b+c),\ y=-\frac{1}{2}(a+2b+c),\ z=-\frac{1}{2}(a+b+2c).$$

17. $$x=\frac{bc-a^2+ca-b^2+ab-c^2}{a(bc-a^2)+b(ca-b^2)+c(ab-c^2)}=\frac{\Sigma bc-\Sigma a^2}{3abc-\Sigma a^3}=\frac{1}{a+b+c}.$$

Similarly $$y=z=\frac{1}{a+b+c}.$$

18. $y+z-x=b+c$, $z+x-y=c+a$ and $x+y-z=a+b$.

Hence $$x+y-z+z+x-y=a+b+c+a;$$

$\therefore\ x=\frac{1}{2}(2a+b+c)$, and similarly $y=\frac{1}{2}(2b+c+a)$ and $z=\frac{1}{2}(2c+a+b)$.

19. $$x=\frac{(b^2-c^2)+(a+b+c)(c-b)}{bc^2-b^2c+a(b^2-c^2)+a^2(c-b)}=\frac{a(c-b)}{(c-b)(b-a)(c-a)}=\frac{a}{(b-a)(c-a)}.$$

Hence $$y=\frac{b}{(c-b)(a-b)} \text{ and } z=\frac{c}{(a-c)(b-c)}.$$

20. $$x=\frac{(a+b+c)(cb-a^2)+(bc+ca+ab)(a-b)+(bc+ca+ab)(a-c)}{cb-a^2+b(a-b)+c(a-c)}$$

$$=\frac{a\{bc+ca+ab-a^2-b^2-c^2\}}{bc+ca+ab-a^2-b^2-c^2}=a.$$

Hence also $y=b$ and $z=c$.

21. $$x=\frac{(a+b+c)(bc-a^2)+(a^2+b^2+c^2)(a-b)+(a^2+b^2+c^2)(a-c)}{bc-a^2+b(a-b)+c(a-c)}$$

$$=\frac{a^3-2a^2(b+c)+a(2b^2+2c^2+bc)-b^3-c^3}{-a^2+a(b+c)-(b^2+c^2-bc)}$$

$$=-a+b+c.$$

$$y=\frac{(a+b+c)(ac-b^2)+(a^2+b^2+c^2)(b-c)+(a^2+b^2+c^2)(b-a)}{ac-b^2+c(b-c)+a(b-a)}$$

$$=\frac{b^3-2b^2(c+a)+b(2c^2+2a^2+ca)-c^3-a^3}{-b^2+b(c+a)-(c^2+a^2-ca)}$$

$$=-b+c+a.$$

Hence, from (i), $z=-c+a+b$.

22. $$x=\frac{(b-c)(c-a)(a-b)(ca-ab)}{ab(c+a)-ca(a+b)+(b+c)(ca-ab)+bc(b-c)}$$

$$=\frac{-a(b-c)^2(c-a)(a-b)}{b^2c-bc^2+c^2a-ca^2+a^2b-ab^2}$$

$$=a(b-c).$$

The values of y and z are found by cyclical changes of a, b, c. Hence

$$x=a(b-c),\ y=b(c-a) \text{ and } z=c(a-b).$$

23. $$x=\frac{a(bc-a^2)+b(ca-b^2)+c(ab-c^2)}{a(bc-a^2)+b(ca-b^2)+c(ab-c^2)}=1.$$

Then $$by+cz=cy+az=ay+bz=0;$$

$$\therefore\ y=z=0.$$

24. It is clear that a, b, c are the three roots of the cubic in λ

$$x-\lambda y+\lambda^2 z-\lambda^3=0.$$

Hence, from Art. 129,

$$a+b+c=z,\ bc+ca+ab=y \text{ and } abc=x.$$

25. $$x=\frac{m(b^2c^3-b^3c^2)+m^2(b^3c-bc^3)+m^3(bc^2-b^2c)}{a(b^2c^3-b^3c^2)+a^2(b^3c-bc^3)+a^3(bc^2-b^2c)}$$

$$=\frac{m\{bc-m(b+c)+m^2\}}{a\{bc-a(b+c)+a^2\}}=\frac{m(m-b)(m-c)}{a(a-b)(a-c)}.$$

The values of y and z will be given by cyclical changes of a, b, c.

26. $$x=\frac{(a^2+2bc)(bc-a^2)+(b^2+2ca)(ab-c^2)+(c^2+2ab)(ca-b^2)}{a(bc-a^2)+c(ab-c^2)+b(ca-b^2)}$$
$$=\frac{a\{3abc-a^3-b^3-c^3\}}{3abc-a^3-b^3-c^3}=a,$$
$$y=\frac{(a^2+2bc)(ab-c^2)+(b^2+2ca)(ca-b^2)+(c^2+2ab)(bc-a^2)}{c(ab-c^2)+b(ca-b^2)+a(bc-a^2)}=b,$$
and $z=c$.

27. $$x=\frac{2(a+b+c)\{b(a-b)-c(c-a)\}+2(bc+ca+ab)(b+c-2a)}{b(a-b)-c(c-a)+a(b+c-2a)+(b-c)(c-b)}$$
$$=\frac{2(b+c)\{-a^2+a(b+c)-b^2+bc-c^2\}}{2\{-a^2+a(b+c)-b^2+bc-c^2\}}=b+c.$$

The values of y and z will be given by cyclical changes of a, b, c.

28. $$x=\frac{(a+b+c)(ab^3-ac^3)+(a+b+c)^2(ac^2-ab^2)}{a(ab^3-ac^3)+a^2(ac^2-ab^2)+bc(bc^2-b^2c)}$$
$$=\frac{-a(a+b+c)(b-c)(bc+ca+ab)}{-(b-c)\{a^3(b+c)-a^2(b^2+bc+c^2)+b^2c^2\}}$$
$$=\frac{a(a+b+c)}{a^2-a(b+c)+bc}=\frac{a(a+b+c)}{(a-b)(a-c)}.$$

The values of y and z will be given by cyclical changes of a, b and c.

29. $$x=\frac{(l+m+n)\{m(l-m)-n(n-l)\}+(mn+nl+lm)(m+n-2l)}{m(l-m)-n(n-l)+l(m+n-2l)+(m-n)(n-m)}$$
$$=\frac{-(m+n)\{l^2-(m+n)l+m^2-mn+n^2\}}{-2\{l^2-(m+n)l+m^2-mn+n^2\}}$$
$$=\frac{1}{2}(m+n).$$

The values of y and z are found by cyclical changes of l, m, n.

30. $$x=\frac{(l^3+m^3+n^3-3lmn)\{mn-l^2+lm-n^2+nl-m^2\}}{l(mn-l^2)+n(lm-n^2)+m(nl-m^2)}$$
$$=l^2+m^2+n^2-mn-nl-lm.$$

Hence we have
$$x=y=z=l^2+m^2+n^2-mn-nl-lm.$$

31. $$x=\frac{(l+m+n)\{m^2n^2-l^2mn+lmn^2-m^3n+lm^2n-mn^3\}}{l^2(m^2n^2-l^2mn)+lm(lmn^2-m^3n)+nl(lm^2n-mn^3)}$$
$$=\frac{mn(l+m+n)(mn+nl+lm-l^2-m^2-n^2)}{lmn(3lmn-l^3-m^3-n^3)}=\frac{1}{l}.$$

Thus
$$lx=my=nz=1.$$

32. Multiply the equations in order by $a+\gamma$, $b+\gamma$, $c+\gamma$ respectively. Then

$$x\left(\frac{a+\gamma}{a+a}-\frac{b+\gamma}{b+a}\right)+y\left(\frac{a+\gamma}{a+\beta}-\frac{b+\gamma}{b+\beta}\right)=a+\gamma-(b+\gamma);$$

$$\therefore \frac{x(a-\gamma)}{(a+a)(b+a)}+\frac{y(\beta-\gamma)}{(a+\beta)(b+\beta)}=1,$$

and similarly

$$\frac{x(a-\gamma)}{(b+a)(c+a)}+\frac{y(\beta-\gamma)}{(b+\beta)(c+\beta)}=1.$$

Multiply these last equations by $a+\beta$ and $c+\beta$ respectively; then we have

$$\frac{x(a-\gamma)}{(a+a)(b+a)(c+a)}\{(c+a)(a+\beta)-(a+a)(c+\beta)\}=a-c;$$

$$\therefore x(a-\gamma)(a-\beta)=(a+a)(b+a)(c+a).$$

The values of y and z can now be written down.

33. Subtract twice the first equation from the sum of the other three; then $3x=b+c+d-2a$, and similarly for y, z and w.

34. It is clear that a, b, c, d are the four roots of the following equation in λ

$$\lambda^4+w\lambda^3+z\lambda^2+y\lambda+x=0.$$

Hence

$$w=-(a+b+c+d),\ z=\Sigma bc,$$

$$y=-\Sigma bcd \text{ and } x=abcd.$$

XIII. Pages 163—165.

1. We have $(23-y)^2-y^2=23$, $\therefore y=11$, and $x=23-y=12$.

2. $y=2x-1$; $\therefore x^2-4(2x-1)^2+x+3(2x-1)=1$; $\therefore 15x^2-23x+8=0$;

$$\therefore x=1 \text{ or } x=\frac{8}{15}.$$

Then $y=2x-1=1$ or $\frac{1}{15}$.

3. We have $x^2+xy=12(xy-2y^2)$; therefore $x-3y=0$ or $x-8y=0$. Hence either $9y^2+3y^2=12$ giving $y=\pm 1$ and $x=\pm 3$, or else $64y^2+8y^2=12$ giving $y=\pm\frac{1}{\sqrt{6}}$ and $x=\pm\frac{8}{\sqrt{6}}$.

4. $17(x^2+2y^2)=22(3y^2-xy-x^2)$; therefore $39x^2+22xy-32y^2=0$, whence $3x=2y$ or $13x=-16y$. Then, either $\left(\frac{4}{9}+2\right)y^2=22$ giving $y=\pm 3$ and $x=\pm 2$, or $\left(\frac{16^2}{13^2}+2\right)y^2=22$ giving $y=\pm\frac{13}{\sqrt{27}}$ and $x=\mp\frac{16}{\sqrt{27}}$.

5. $x-y=5$ and $x-y=\frac{5xy}{84}$; therefore $xy=84$. Then $(x-y)^2+4xy=361$, whence $x+y=\pm 19$. Hence $2x=5\pm 19$ and $2y=\pm 19-5$.

6. Multiply the second equation by $(x+b)(y+a)$ and then substitute $a+b-x$ for y; we then have after simplification $x^2-(3a-b)x+2a^2-ab=0$, whence $x=a$ or $x=2a-b$. Then $y=a+b-x=b$ or $2b-a$.

7. We have $x(b-a)=(a+b)y$ and $a(x+y)=xy$;

therefore $$a\{(a+b)y+(b-a)y\}=(a+b)y^2.$$

Hence $y=0$ or $y=2ab/(a+b)$; then $x=0$ or $x=2ab/(b-a)$.

8. By addition $$\left(\frac{1}{x}+\frac{1}{y}\right)^2=\frac{1}{a^2}+\frac{1}{b^2};$$

$$\therefore \frac{1}{x}+\frac{1}{y}=\pm\frac{1}{ab}\sqrt{(a^2+b^2)}.$$

Hence $x=a^2\left(\frac{1}{x}+\frac{1}{y}\right)=\pm\frac{a}{b}\sqrt{(a^2+b^2)}$, and similarly $y=\pm\frac{b}{a}\sqrt{(a^2+b^2)}$.

9. We have $$\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{2ab}{xy}=16 \text{ and } \frac{a^2}{x^2}+\frac{b^2}{y^2}-\frac{2ab}{xy}=4.$$

Hence $$\frac{a}{x}+\frac{b}{y}=\pm 4 \text{ and } \frac{a}{x}-\frac{b}{y}=\pm 2,$$

whence $$2\frac{a}{x}=\pm 4\pm 2=\pm 6 \text{ or } \pm 2 \text{ and } 2\frac{b}{y}=\pm 4\mp 2=\pm 2 \text{ or } \pm 6.$$

The pairs of corresponding values of x and y are therefore $\frac{a}{3}, b$; $-\frac{a}{3}, -b$;

$$a, \frac{b}{3}; \ -a, -\frac{b}{3}.$$

10. Divide the members of the second equation by the corresponding members of the first; then we have

$$x^2-xy+y^2=\frac{b^3}{a}.$$

Also $x^2+2xy+y^2=4a^2$. Hence $3xy=4a^2-\frac{b^3}{a}$, and therefore

$$(x-y)^2=\frac{b^3}{a}-\frac{1}{3}\left(4a^2-\frac{b^3}{a}\right),$$

so that $$x-y=\pm\sqrt{\frac{4b^3-4a^3}{3a}}.$$

Hence $$2x=2a\pm\sqrt{\frac{4b^3-4a^3}{3a}}, \text{ and } 2y=2a\mp\sqrt{\frac{4b^3-4a^3}{3a}}.$$

11. $x^2+xy+y^2=(x^4+x^2y^2+y^4)/(x^2-xy+y^2)=4251/109=39.$

Hence $2xy=39-109=-70$. Then $(x+y)^2=4$ and $(x-y)^2=144$, &c.

12. $x-\sqrt{xy}+y=(x^2+xy+y^2)/(x+\sqrt{xy}+y)=133\div19=7.$

Hence $2\sqrt{xy}=12$, and therefore $(\sqrt{x}-\sqrt{y})^2=1$ and $(\sqrt{x}+\sqrt{y})^2=25$. Hence $\sqrt{x}-\sqrt{y}=\pm1$ and $\sqrt{x}+\sqrt{y}=\pm5$, so that $2\sqrt{x}=\pm1\pm5$ giving $x=9$ or 4, and similarly $y=4$ or 9.

13. Cube the second equation; then $x+3\sqrt[3]{x}\sqrt[3]{y}(\sqrt[3]{x}+\sqrt[3]{y})+y=216$.

Hence $18\sqrt[3]{x}\sqrt[3]{y}=216-72$, whence $xy=512$. Then

$$(x-y)^2=(x+y)^2-4xy=5184-2048;\ \therefore\ x-y=\pm56.$$

Hence $$2x=72\pm56 \text{ and } 2y=72\mp56.$$

14. By subtraction $xy=6$; and then $x+y=2xy=12$. Hence

$$(x-y)^2=144-24, \text{ giving } x-y=\pm2\sqrt{30}.$$

Then $$2x=12\pm2\sqrt{30} \text{ and } 2y=12\mp2\sqrt{30}.$$

15. Substitute for y; then $x^5+(1-x)^5=31$, whence

$$x^4-2x^3+2x^2-x-6=0, \text{ or } (x+1)(x^3-3x^2+5x-6)=0,$$

or $$(x+1)(x-2)(x^2-x+3)=0.$$

Hence $$x=-1 \text{ or } x=2 \text{ or } x=\frac{1}{2}(1\pm\sqrt{-11}),$$

the corresponding values of y being

$$y=2 \text{ or } y=-1 \text{ or } y=\frac{1}{2}(1\mp\sqrt{-11}).$$

Or thus:

Put $$x=\frac{1}{2}+z;$$ then, from the first equation, $y=\frac{1}{2}-z$.

Hence $$\left(\frac{1}{2}+z\right)^5+\left(\frac{1}{2}-z\right)^5=31, \text{ whence } \frac{1}{2^4}+\frac{10}{2^2}z^2+5z^4=31$$

or $$16z^4+8z^2=99.$$

Hence $4z^2=9$ or -11 and therefore $z=\pm\frac{3}{2}$ or $\pm\frac{1}{2}\sqrt{-11}$.

16. Subtract the second equation from the first; then

$$(x+y)^2-6(x+y)+8=0,$$

whence $$x+y=4 \text{ or } x+y=2.$$

Then $$16+xy-16+3=0 \text{ or else } 4+xy-8+3=0,$$

so that $$xy=-3 \text{ or else } xy=1.$$

Then $x+y=4$ and $xy=-3$

give $x=2\pm\sqrt{7}$ and $y=2\mp\sqrt{7}$;

also $x+y=2$ and $xy=1$

give $x=1$ and $y=1$.

17. By addition $(x+y)^2+(x+y)-42=0$,

whence $x+y+7=0$ or $x+y-6=0$.

Then, from $x(x+y+1)=14$,

we have $-6x=14$ or $7x=14$;

therefore $x=-\frac{7}{3}$ or $x=2$,

the corresponding values of y being

$$y=-\frac{14}{3} \text{ or } y=4.$$

18. $x+y=(x^3+y^3)/(x^2-xy+y^2)=3$.

Then, since $x^2+2xy+y^2=9$ and $x^2-xy+y^2=3$,

we have $xy=2$ and therefore $(x-y)^2=1$.

Thus $x+y=3$ and $x-y=\pm 1$.

19. We have $ay=bx$ and $xy-bx=2ab$;

$$\therefore\quad bx^2-abx=2a^2b,$$

whence $x=2a$ or $x=-a$,

the corresponding values of y being $2b$ and $-b$.

20. $x-1=-\frac{1}{y}$ and $4x-1=yx$;

$$\therefore\quad (x-1)(4x-1)=-x, \text{ that is } (2x-1)^2=0.$$

Hence $x=\frac{1}{2}$ and then $y=2$.

21. We have $ax+by=2ab$ and $ax+by=2xy$;

$$\therefore\quad xy=ab.$$

Hence $(ax-by)^2=(ax+by)^2-4abxy=0$.

Then, from $ax+by=2ab$ and $ax-by=0$,

we have $x=b,\ y=a$.

22. $x^3+y^3=12xy$ and $x+y=\frac{1}{3}xy$;

$$\therefore \quad x^2-xy+y^2=36.$$

Hence $(x+y)^2-36=3xy=9(x+y)$;

whence $x+y=12$ or -3, and therefore $xy=36$ or -9.

Then $x^2-2xy+y^2=0$ or 45, giving $x-y=0$ or $\pm\sqrt{45}$.

We have now $x+y=12$, $x-y=0$ giving $x=y=6$;

or $x+y=-3$, $x-y=\pm\sqrt{45}$

giving $x=\frac{1}{2}(-3\pm\sqrt{45})$ and $y=\frac{1}{2}(-3\mp\sqrt{45})$.

23. We have

$$\frac{x}{y}(x^2+y^2)=a^2 \text{ and } \frac{y}{x}(x^2+y^2)=b^2.$$

Hence, by division and multiplication, we have

$$(x^2+y^2)^2=a^2b^2 \text{ and } \frac{x^2}{y^2}=\frac{a^2}{b^2}, \text{ \&c.}$$

24. $$(xy-a)\left(xy-\frac{1}{a}\right)=\frac{x}{y}\cdot\frac{y}{x};$$

$$\therefore \quad x^2y^2-\left(a+\frac{1}{a}\right)xy=0,$$

so that $xy=0$ or $xy=a+\frac{1}{a}$.

If $xy=0$ it follows that x and y must both be zero but $x:y=-a$.

If $xy=a+\frac{1}{a}$, we have $\frac{x}{y}=\frac{1}{a}$, and therefore

$$x^2=\frac{a^2+1}{a^2} \text{ and } y^2=a^2+1.$$

25. Multiply the equations by x, x^2 respectively; then

$$x^2+xy+y^2=14x \text{ and } x^4+x^2y^2+y^4=84x^2.$$

Hence, by division, $x^2-xy+y^2=6x$.

We then have $2xy=8x$, and therefore $x=0$ or $y=4$.

But $x=0$ cannot be a root of the original equation for this would require that $y=0$ and that $14^2=84$. Hence $y=4$, from which it follows that $x=8$ or $x=2$.

26. $x^2+y^2=(x+y)^2-2xy$ and $x^3+y^3=(x+y)\{(x+y)^2-3xy\}$;

$$\therefore \quad 6(36-2xy)(36-3xy)=1440,$$

whence $x^2y^2-30xy+176=0$, giving $xy=8$ or $xy=22$.

Then $(x-y)^2=(x+y)^2-4xy=4$ or -52,

so that $x-y=\pm 2$ or $\pm\sqrt{-52}$,

and therefore $2x=8$ or 4 or $6\pm\sqrt{-52}$

and $2y=4$ or 8 or $6\mp\sqrt{-52}$.

27. Square the members of the first equation, and subtract the second; then $2xy=24x^2y^2$, hence $xy=0$ which gives $x=0$ *and* $y=0$, or $xy=\frac{1}{12}$.

From $xy=\frac{1}{12}$ and $x+y=8xy=\frac{2}{3}$,

we have $x=\frac{1}{2}$, $y=\frac{1}{6}$ or $x=\frac{1}{6}$, $y=\frac{1}{2}$.

28. Subtract; then $(x-y)^2=8(x-y)+9$,

whence $x-y=-1$ or $x-y=9$.

29. $x(1+3y)=3-y$, and $x(3-y)=1+3y$;

$$\therefore \quad (1+3y)^2=(3-y)^2,$$

whence $y=-2$ or $y=\frac{1}{2}$.

30. If $x^2-y^2=0$ then $x-y=0$ and also $x+y=0$, and therefore $x=y=0$. But if x^2-y^2 be not zero, we must have

$$x+y=\frac{1}{a} \text{ and } x-y=\frac{1}{b}.$$

31. Square the first and subtract the second; then $2\frac{xy}{ab}=2$. Hence

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2=\left(\frac{b}{a}-\frac{a}{b}\right)^2, \quad \therefore \quad \frac{x}{a}-\frac{y}{b}=\pm\left(\frac{b}{a}-\frac{a}{b}\right).$$

Hence $\frac{x}{a}=\frac{b}{a}$ or $\frac{a}{b}$ and $\frac{y}{b}=\frac{a}{b}$ or $\frac{b}{a}$.

32. We have $by(x^2+a^2)-ax(y^2+b^2)=0$,

that is $(xy-ab)(bx-ay)=0$.

Hence $xy=ab$ or $bx=ay$.

If $xy=ab$, we have $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{x^2}{ab}+\dfrac{ab}{x^2}$,

and therefore $x=b$ or $x^3=a^2b$,

and then $y=a$ or $y^3=ab^2$.

Again, if $\dfrac{x}{a}=\dfrac{y}{b}$, we have $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{a}{b}+\dfrac{b}{a}$,

whence $x=b$ or $x=\dfrac{a^2}{b}$,

the corresponding values of y being

$$y=\frac{b^2}{a} \text{ and } y=a \text{ respectively.}$$

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1. We have $\dfrac{zx\,.\,xy}{yz}=\dfrac{b^2c^2}{a^2}$; $\therefore\ ax=\pm bc$,

and similarly $by=\pm ca$ and $cz=\pm ab$.

But, since $yz=a^2$, y and z must *both* be positive or *both* negative; we therefore have $a^2x=b^2y=c^2z=\pm abc$.

2. By addition $(x+y+z)^2=a^2+b^2+c^2$.

Hence $\dfrac{x}{a^2}=\dfrac{y}{b^2}=\dfrac{z}{c^2}=\pm\dfrac{1}{\sqrt{(a^2+b^2+c^2)}}$.

3. Adding the second and third we have

$$yz=\frac{1}{2}(b+c),$$

and similarly $zx=\dfrac{1}{2}(c+a)$ and $xy=\dfrac{1}{2}(a+b)$.

Hence, as in question 1, we have

$$(b+c)x=(c+a)y=(a+b)z=\pm\sqrt{\left\{\frac{1}{2}(b+c)(c+a)(a+b)\right\}}.$$

4. It is obvious that $x=y=z=0$ satisfies the equations; and that if $x=0$ then y and z must also be zero. If xyz is not zero, divide the equations in order by ayz, bzx and cxy respectively; we then have

$$\frac{1}{a}=\frac{1}{y}+\frac{1}{z},\quad \frac{1}{b}=\frac{1}{z}+\frac{1}{x},\quad \frac{1}{c}=\frac{1}{x}+\frac{1}{y}.$$

Hence $\dfrac{2}{x}=\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}$, &c.

5. It is obvious that the equation is satisfied by $x=y=z=0$; and that if x or y or z is zero, then the other two must also be zero. If $xyz \neq 0$, divide the equations in order by $bcyz$, $cazx$ and $abxy$ respectively; then

$$\frac{1}{bc}=\frac{1}{cz}+\frac{1}{by}, \quad \frac{1}{ca}=\frac{1}{ax}+\frac{1}{cz}, \quad \frac{1}{ab}=\frac{1}{by}+\frac{1}{ax}.$$

Hence
$$-\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{2}{ax},$$

whence $x=\dfrac{2bc}{b+c-a}$, and so for y and z.

6. Subtract the second equation from the first; then

$$(x-y)(x+y-2z)=0; \quad \therefore \text{ (i) } x-y=0 \text{ or (ii) } x+y-2z=0.$$

Subtract the third equation from the second; then

$$(y-z)(y+z-2x)=0; \quad \therefore \text{ (iii) } y-z=0 \text{ or (iv) } y+z-2x=0.$$

Now it is easily seen that (i) and (iii), (i) and (iv), (ii) and (iii), and (ii) and (iv) all lead to $x=y=z$; and substituting in either of the given equations we have $x=y=z=\pm 2$.

7. By addition
$$2(x+y+z)^2=a+b+c.$$

Also
$$2x(x+y+z)=b+c-a.$$

Hence
$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=\pm\frac{1}{\sqrt{(2a+2b+2c)}}.$$

8. We have
$$(x+a)^2=\frac{b^2c^2}{a^2}; \quad \therefore \quad x+a=\pm\frac{bc}{a},$$

and similarly
$$y+b=\pm\frac{ca}{b} \text{ and } z+c=\pm\frac{ab}{c},$$

and it is obvious that the signs must all be positive or all negative.

Hence
$$x=a-\pm\frac{bc}{a}, \quad y=-b\pm\frac{ca}{b} \text{ and } z=-c\pm\frac{ab}{c}.$$

9. We have
$$(x+y-z)(x-y+z)=a^2 \quad \text{..... (i)},$$
$$(-x+y+z)(x+y-z)=b^2 \quad \text{..... (ii)},$$
and
$$(x-y+z)(-x+y+z)=c^2 \quad \text{..... (iii)}.$$

Hence
$$(-x+y+z)^2=\frac{b^2c^2}{a^2},$$

and therefore
$$-x+y+z=\pm\frac{bc}{a}.$$

Similarly
$$x-y+z=\pm\frac{ca}{b},$$

and $$x+y-z=\pm\frac{ab}{c},$$

all the signs in the ambiguities being positive or all negative.

Hence $$x=\pm\frac{1}{2}\left(\frac{ca}{b}+\frac{ab}{c}\right)=\pm\frac{a}{2bc}(b^2+c^2),$$

and so for y and z.

10. Take the first equation from the sum of the other two; then we have $$x^2-(y-z)^2=b+c-a.$$

We have similarly

$$y^2-(z-x)^2=c+a-b \text{ and } z^2-(x-y)^2=a+b-c.$$

Hence, from 9, we have

$$x=\pm\frac{a\sqrt{(b+c-a)}}{\sqrt{(c+a-b)}\sqrt{(a+b-c)}}.$$

11. Since $$\frac{y+z}{a}=\frac{z+x}{b}=\frac{x+y}{c},$$

each fraction $$=\frac{2x}{b+c-a}=\frac{2y}{c+a-b}=\frac{2z}{a+b-c}.$$

Hence $$yz=\frac{1}{b+c-a}, \text{ \&c.}$$

Hence $$x^2=\frac{b+c-a}{(c+a-b)(a+b-c)}, \text{ \&c.}$$

12. Since $$\frac{y+z}{a}=\frac{z+x}{b}=\frac{x+y}{c},$$

each fraction

$$=\frac{2x}{b+c-a}=\frac{2y}{c+a-b}=\frac{2z}{a+b-c}=\frac{2\sqrt{(x^2+y^2+z^2)}}{\sqrt{(3\Sigma a^2-2\Sigma bc)}}.$$

Hence $$\sqrt{(x^2+y^2+z^2)}=\frac{2(a^2+b^2+c^2)}{\sqrt{(3\Sigma a^2-2\Sigma bc)}},$$

unless $\sqrt{(x^2+y^2+z^2)}=0$ which requires that $x=y=z=0$.

Hence $x=y=z=0$, or

$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=\frac{2(a^2+b^2+c^2)}{3\Sigma a^2-2\Sigma bc}.$$

13. The equations may be written

$$(y-1)(z-1)=a+1,\ (z-1)(x-1)=b+1,\ (x-1)(y-1)=c+1.$$

Hence, as in 1, we have

$$x=1\pm\sqrt{(b+1)}\sqrt{(c+1)}/\sqrt{(a+1)}, \text{ \&c.}$$

14. We have

$$(y-a)(z-a)=a^2+\alpha,\ (z-a)(x-a)=a^2+\beta,\ (x-a)(y-a)=a^2+\gamma.$$

Hence, as in **1**, we have

$$x=a\pm\sqrt{(a^2+\beta)}\sqrt{(a^2+\gamma)}/\sqrt{(a^2+\alpha)},\ \&c.$$

15. We have

$$(y-b)(z-c)=f^2+bc,\ (z-c)(x-a)=g^2+ca,\ (x-a)(y-b)=h^2+ab.$$

Hence, as in **1**, we have

$$x=a\pm\sqrt{(g^2+ca)}\sqrt{(h^2+ab)}/\sqrt{(f^2+bc)},\ \&c.$$

16. From the first two equations we have

$$x=\frac{3y-2}{2y}\ \text{and}\ z=\frac{3}{7-3y};$$

hence from the third equation

$$\frac{3}{7-3y}+\frac{2y}{3y-2}=4\ \text{giving}\ y=2\ \text{or}\ y=\frac{5}{6}.$$

Hence $$x=\frac{3}{2}-\frac{1}{y}=1\ \text{or}\ \frac{3}{10},$$

and $$z=\frac{3}{7-3y}=3\ \text{or}\ \frac{2}{3}.$$

Hence the values are $1, 2, 3;\ \frac{3}{10}, \frac{5}{6}, \frac{2}{3}$.

17. We have

$$36=(x+y+z)^2=x^2+y^2+z^2+2yz+2zx+2xy=14+2yz+2zx+2xy;$$
$$\therefore\ yz+zx+xy=11.$$

Hence x, y, z are roots of the cubic

$$\lambda^3-6\lambda^2+11\lambda-6=0.$$

Now $\lambda=1$ clearly satisfies the cubic: we therefore have

$$(\lambda-1)(\lambda^2-5\lambda+6)=0\ \text{or}\ (\lambda-1)(\lambda-2)(\lambda-3)=0.$$

Hence the values of x, y, z, taken in any order, are 1, 2, 3 respectively.

18. $x^3+y^3+z^3-3xyz=(x+y+z)\{(x+y+z)^2-3yz-3zx-3xy\}$.

Hence $495-315=15\{225-3\Sigma yz\}$, whence $\Sigma yz=71$.

Hence x, y, z are the three roots of the cubic

$$\lambda^3-15\lambda^2+71\lambda-105=0.$$

By inspection we find that the cubic is satisfied by $\lambda=3$; the other roots are therefore given by $\lambda^2-12\lambda+35=0$, that is $(\lambda-5)(\lambda-7)=0$.

Hence the values of x, y, z, taken in any order, are 3, 5, 7 respectively.

19. $2\Sigma yz = (x+y+z)^2 - (x^2+y^2+z^2) = 81 - 41.$

Also $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-yz-zx-xy);$

$$\therefore\ 189 - 3xyz = 9(41-20);\ \therefore\ xyz = 0.$$

Hence x, y, z are the three roots of the cubic

$$\lambda^3 - 9\lambda^2 + 20\lambda = 0,$$

that is $$\lambda(\lambda-4)(\lambda-5) = 0.$$

Hence the values of x, y, z, taken in any order, are 0, 4, 5 respectively.

20. $(y+z)(z+x)(x+y) = (10-x)(10-y)(10-z)$

$$= 1000 - 100(x+y+z) + 10(yz+zx+xy) - xyz$$

$$= 330 - xyz = 294;\ \therefore\ xyz = 36.$$

Hence x, y, z are the roots of the cubic

$$\lambda^3 - 10\lambda^2 + 33\lambda - 36 = 0.$$

By inspection we find that the cubic is satisfied by $\lambda = 3$. Hence the cubic is

$$(\lambda-3)(\lambda^2-7\lambda+12) = 0,$$

that is $$(\lambda-3)(\lambda-3)(\lambda-4) = 0.$$

Hence the values of x, y, z, taken in any order, are 3, 3, 4 respectively.

21. The equations are satisfied by the values $x=y=z=0$; and, if either of the three is zero, the others must also be zero.

Now, if $xyz \neq 0$, the equations may be written

$$\frac{b}{y}+\frac{c}{z} = \frac{c}{z}+\frac{a}{x} = \frac{a}{x}+\frac{b}{y} = \frac{a^2+b^2+c^2}{x^2+y^2+z^2}.$$

Hence $$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

and therefore each $$= \frac{\sqrt{(a^2+b^2+c^2)}}{\sqrt{(x^2+y^2+z^2)}};$$

$$\therefore\ \frac{2\sqrt{(a^2+b^2+c^2)}}{\sqrt{(x^2+y^2+z^2)}} = \frac{a^2+b^2+c^2}{x^2+y^2+z^2},$$

so that $$\sqrt{(a^2+b^2+c^2)} = 2\sqrt{(x^2+y^2+z^2)},$$

and therefore $$\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = 2.$$

22. Subtract the second from the first;

then $$\frac{a}{x} - \frac{x}{a} + \frac{y}{b} - \frac{b}{y} = 0;\ \therefore (ay - bx)(xy + ab) = 0.$$

Hence (i) $\frac{x}{a}=\frac{y}{b}$ or (ii) $\frac{x}{a}+\frac{b}{y}=0$.

(i) If $\frac{x}{a}=\frac{y}{b}$ we have $\frac{z}{c}=1-\frac{x}{a}-\frac{a}{x}$, and $\frac{c}{z}=1-2\frac{x}{a}$.

Hence $$1=\left(1-2\frac{x}{a}\right)\left(1-\frac{x}{a}-\frac{a}{x}\right),$$

whence $2x^3-3ax^2+2a^2x-a^3=0$, or $(x-a)(2x^2-ax+a^2)=0$.

Hence $$x=a \text{ or } x=\frac{a}{4}(1\pm\sqrt{-7}).$$

If $x=a$, $y=b$, and therefore from $\frac{a}{x}+\frac{y}{b}+\frac{z}{c}=1$, $z=-c$.

If $x=\frac{a}{4}(1\pm\sqrt{-7})$, $y=\frac{b}{4}(1\pm\sqrt{-7})$ and therefore $z=\frac{c}{4}(1\pm\sqrt{-7})$.

(ii) If $\frac{x}{a}+\frac{b}{y}=0$, then from the second equation $z=c$.

Hence, from the third equation, $\frac{x}{a}+\frac{y}{b}=0$; $\therefore \frac{b}{y}=\frac{y}{b}$, so that $y=\pm b$. Hence $x=\mp a$.

The simultaneous values of x, y, z are therefore $a, -b, c$; $-a, b, c$; $a, b, -c$;

$$\frac{a}{4}(1\pm\sqrt{-7}),\ \frac{b}{4}(1\pm\sqrt{-7}),\ \frac{c}{4}(1\pm\sqrt{-7}).$$

23. We have $axyz=y^2+z^2$, $bxyz=z^2+x^2$, $cxyz=x^2+y^2$;

$$\therefore (b+c-a)\,xyz=2x^2, \text{ \&c.}$$

Hence $$(b+c-a)(c+a-b)(a+b-c)\,xyz=8;$$

$$\therefore 2x^2(c+a-b)(a+b-c)=8; \text{ \&c.}$$

24. We have $y^2-zx+z^2-xy=a^2$, &c.

Hence $$x^2-yz=\frac{1}{2}(b^2+c^2-a^2), \text{ \&c.,}$$

which is the same as Ex. 5, Art. 151.

Or thus:

By addition $(y-z)^2+(z-x)^2+(x-y)^2=a^2+b^2+c^2$.

From the first two equations

$$y^2-x^2+z(y-x)=a^2-b^2;$$

$$\therefore (y-x)(x+y+z)=(a^2-b^2).$$

Hence $$\frac{y-z}{b^2-c^2}=\frac{z-x}{c^2-a^2}=\frac{x-y}{a^2-b^2}=\frac{\sqrt{\{(y-z)^2+(z-x)^2+(x-y)^2\}}}{\sqrt{\{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2\}}}$$

$$=\frac{\sqrt{(\Sigma a^2)}}{\sqrt{[2\Sigma a^4-2\Sigma b^2c^2]}}=\lambda.$$

Multiply the original equations in order by x, y, z respectively; then

$$a^2x+b^2y+c^2z=0.$$

Hence $$a^2x+b^2\{x-(a^2-b^2)\lambda\}+c^2\{x+(c^2-a^2)\lambda\}=0;$$

$$\therefore\ x(a^2+b^2+c^2)=\{a^2(b^2+c^2)-b^4-c^4\}\lambda.$$

Hence $$\frac{x}{a^2(b^2+c^2)-b^4-c^4}=\frac{1}{\sqrt{(\Sigma a^2)}\sqrt{\{2\Sigma a^4-2\Sigma b^2c^2\}}}=\&c.$$

25. From $$x^2+yz-a^2=\frac{1}{2}(x^2+y^2+z^2),$$

we have $$x^2-(y-z)^2=2a^2.$$

We have similarly

$$y^2-(z-x)^2=2b^2,\ \text{and}\ z^2-(x-y)^2=2c^2.$$

Hence $$(-x+y+z)^2=2\frac{b^2c^2}{a^2},\ \&c.;$$

$$\therefore\ -x+y+z=\pm\sqrt{2}\frac{bc}{a},$$

$$x-y+z=\pm\sqrt{2}\frac{ca}{b},$$

and $$x+y-z=\pm\sqrt{2}\frac{ab}{c}.$$

Hence $$x=\pm\frac{a}{\sqrt{2}}\left(\frac{b}{c}+\frac{c}{b}\right),\ \&c.$$

26. Adding the second and third equations we have

$$yz-x^2=\frac{1}{2}(b+c),$$

and similarly $$zx-y^2=\frac{1}{2}(c+a),\ xy-z^2=\frac{1}{2}(a+b);$$

$$\therefore\ (x^2-yz)^2-(y^2-zx)(z^2-xy)=\frac{1}{4}\{(b+c)^2-(c+a)(a+b)\},$$

that is $$x(x^3+y^3+z^3-3xyz)=\frac{1}{4}\{(b+c)^2-(c+a)(a+b)\}.$$

Hence $$\frac{x}{(b+c)^2-(c+a)(a+b)}=\frac{y}{(c+a)^2-(a+b)(b+c)}$$

$$=\frac{z}{(a+b)^2-(b+c)(c+a)},$$

and each fraction

$$= \sqrt{\frac{x^2 - yz}{\{(b+c)^2 - (c+a)(a+b)\}^2 - \{(c+a)^2 - (a+b)(b+c)\}\{(a+b)^2 - (b+c)(c+a)\}}}$$

$$= \sqrt{\frac{-b-c}{2(b+c)\{(b+c)^3 + (c+a)^3 + (a+b)^3 - 3(b+c)(c+a)(a+b)\}}}$$

$$= \sqrt{\frac{-1}{4(a+b+c)\{(b-c)^2 + (c-a)^2 + (a-b)^2\}}}$$

$$= \frac{1}{2\sqrt{(3abc - a^3 - b^3 - c^3)}}.$$

27. Put $x = a+\lambda$, $y = b+\mu$, $z = c+\nu$; then $\lambda+\mu+\nu = 0$,

$$\lambda^2 + \mu^2 + \nu^2 + 2a\lambda + 2b\mu + 2c\nu = 0,$$

$$(b-c)\lambda + (c-a)\mu + (a-b)\mu = 0.$$

From the first and third equations, we have

$$\frac{\lambda}{2a-b-c} = \frac{\mu}{2b-c-a} = \frac{\nu}{2c-a-b}.$$

Hence from the second equation,

$$\lambda^2\{(2a-b-c)^2 + (2b-c-a)^2 + (2c-a-b)^2\}$$
$$+ 2\lambda\{(2a-b-c)\{a(2a-b-c) + b(2b-c-a) + c(2c-a-b)\} = 0,$$

or
$$6\lambda^2(\Sigma a^2 - \Sigma bc) + 4\lambda(2a-b-c)(\Sigma a^2 - \Sigma bc) = 0.$$

Hence
$$\lambda = 0 \text{ or } \lambda = -\frac{2}{3}(2a-b-c).$$

Hence $x = a$, $y = b$, $z = c$; or $x = \frac{1}{3}(2b+2c-a)$, $y = \frac{1}{3}(2c+2a-b)$,

$$z = \frac{1}{3}(2a+2b-c).$$

28. The equations are satisfied by the values $x = y = z = 0$; also, if either of the three quantities x, y, z is zero, it is easily seen that they must all be zero.

Now suppose that $xyz \neq 0$; then we have

$$bcyz = (y+z)^2(z+x)(x+y) = ax(y+z)^2;$$

$$\therefore \frac{bc}{a}xyz = x^2(y+z)^2.$$

Hence
$$x(y+z) = \pm\frac{1}{a}\sqrt{(abcxyz)},$$

$$y(z+x) = \pm\frac{1}{b}\sqrt{(abcxyz)},$$

and
$$z(x+y) = \pm\frac{1}{c}\sqrt{(abcxyz)},$$

all the signs being positive or all negative.

Hence $$2yz = \pm\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\sqrt{(abcxyz)},\ \&c.$$

Hence $$8x^2y^2z^2 = \pm\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\sqrt{(a^3b^3c^3x^3y^3z^3)},$$

or $$8\sqrt{abcxyz} = \pm\, a^2b^2c^2\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right),$$

since $xyz \neq 0$.

Hence $$64\,xyz = a^3b^3c^3\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)^2\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)^2\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)^2,$$

and $$16yz = \pm\, a^2b^2c^2\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)^2\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right);$$

$$\therefore\ 4x = \pm\, abc\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right),\ \&c.$$

29. The equations are clearly satisfied if x, y, z are all zero. Also, if $x=0$, we must have $y=0$ and $z=0$, or $y=0$ and $z=c$, or $y=b$ and $z=0$.

Hence, if $xyz=0$, we have the following sets of values 0, 0, 0; a, 0, 0; 0, b, 0; and 0, 0, c.

Now suppose that xyz is not zero. Multiply the given equations in order by y, z, x and add; and also by z, x, y and add; then we have

$$axy + byz + czx = 0,$$

and $$azx + bxy + cyz = 0.$$

Hence $$\frac{yz}{bc-a^2} = \frac{zx}{ca-b^2} = \frac{xy}{ab-c^2};$$

or, since xyz is not zero,

$$x(bc-a^2) = y(ca-b^2) = z(ab-c^2) = \lambda \text{ suppose.}$$

Hence, by substitution, we have

$$\lambda^2\left\{\frac{1}{(bc-a^2)^2} - \frac{1}{(ca-b^2)(ab-c^2)}\right\} - \frac{a\lambda}{bc-a^2} = 0.$$

Hence $$\lambda = 0 \text{ giving } x=y=z=0;$$

or $$\lambda = \frac{a(bc-a^2)(ca-b^2)(ab-c^2)}{(ca-b^2)(ab-c^2)-(bc-a^2)^2},$$

whence $$x = \frac{(ca-b^2)(ab-c^2)}{3abc-a^3-b^3-c^3},\ \&c.$$

30. We have

$$a(2x+y+z) = (x+y+z)^2 - x^2 = (2x+y+z)(y+z).$$

Hence $$2x+y+z=0 \text{ or } y+z=a.$$

So also $$2y+z+x=0 \text{ or } z+x=b,$$

and $$2z+x+y=0 \text{ or } x+y=c.$$

Now $2x+y+z=0$, $2y+z+x=0$ and $2z+x+y=0$ give $x=y=z=0$.

Again $2y+z+x=0$, $2z+x+y=0$ and $y+z=a$,

give $$x=-\frac{3}{2}a,\ y=\frac{a}{2},\ z=\frac{a}{2};$$

$2z+x+y=0$, $2x+y+z=0$ and $z+x=b$,

give $$x=\frac{b}{2},\ y=-\frac{3}{2}b,\ z=\frac{b}{2};$$

and $2x+y+z=0$, $2y+z+x=0$ and $x+y=c$,

give $$x=\frac{c}{2},\ y=\frac{c}{2},\ z=-\frac{3}{2}c.$$

Also $2x+y+z=0$, $z+x=b$ and $x+y=c$ are inconsistent; and so also are the other two corresponding sets of equations.

Finally $y+z=a$, $z+x=b$ and $x+y=c$, give

$$x=\frac{1}{2}(-a+b+c),\ y=\frac{1}{2}(a-b+c),\ z=\frac{1}{2}(a+b-c).$$

31. By addition we have

$$2(x+y+z)^2-3(yz+zx+xy)=a^2+b^2+c^2 \quad \text{..... (i).}$$

From the last two equations we have $(z-y)(x+y+z)=b^2-c^2$. Hence

$$\frac{y-z}{b^2-c^2}=\frac{z-x}{c^2-a^2}=\frac{x-y}{a^2-b^2}=-\frac{1}{x+y+z}.$$

Hence $$\frac{(y-z)^2+(z-x)^2+(x-y)^2}{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}=\frac{1}{(x+y+z)^2};$$

$$\therefore\ (x+y+z)^4-3(yz+zx+xy)(x+y+z)^2=\Sigma a^4-\Sigma b^2c^2 \quad \text{..... (ii).}$$

From (i) and (ii)

$$\frac{9}{4}(yz+zx+xy)^2=\frac{1}{4}(a^2+b^2+c^2)^2-(\Sigma a^4-\Sigma b^2c^2);$$

$$\therefore\ 3(yz+zx+xy)^2=2\Sigma b^2c^2-\Sigma a^4 \quad \text{.....(iii).}$$

Hence from (i)

$$x+y+z=\sqrt{\left[\frac{1}{2}(a^2+b^2+c^2)+\frac{1}{2}\sqrt{(6\Sigma b^2c^2-3\Sigma a^4)}\right]}=\lambda \text{ suppose.}$$

Then $$y=x+\frac{a^2-b^2}{\lambda};$$

and by substituting in the third of the given equations we have a quadratic equation to find x, and similarly for y and z.

32. We have

$$(a^2x+b^2y)\left\{\frac{(b-c)^2}{ax}+\frac{(c-a)^2}{by}\right\}=c^2z\,.\,\frac{(a-b)^2}{cz}=c\,(a-b)^2\,;$$

$\therefore\ a^3\,(c-a)^2x^2+b^3\,(b-c)^2\,y^2+abxy\,\{a\,(b-c)^2+b\,(c-a)^2-c\,(a-b)^2\}=0,$

that is $a^3\,(c-a)^2x^2+b^3\,(b-c)^2y^2+ab\,(a+b)\,(a-c)\,(b-c)\,xy=0\,;$

$\therefore\ \{a^2\,(c-a)\,x-b^2\,(b-c)\,y\}\,\{a\,(c-a)\,x-b\,(b-c)\,y\}=0.$

Hence $$\frac{a^2x}{b-c}=\frac{b^2y}{c-a}=\text{similarly }\frac{c^2z}{a-b}=\lambda\,;$$

or else $$\frac{ax}{b-c}=\frac{by}{c-a}=\text{similarly }\frac{cz}{a-b}=\mu.$$

Substitute in the third equation; then it will be found that

$$\lambda^2=-abc,\ \text{and}\ \ \mu^2=-\frac{abc}{bc+ca+ab}.$$

XV. Pages 177—181.

1. Let x be the greater part, then $50-x$ will be the smaller, and we have

$$2x=3\,(50-x).$$

2. Let x be the number of pounds A has, then B will have $x+5$, and C will have $x+x+5$; then $x+(x+5)+(2x+5)=50$.

3. Let x be the number of years ago; then the ages at that time were $70-x$ and $45-x$ respectively; hence we have $70-x=2\,(45-x)$.

4. Let x shillings be the original price per score, then $\frac{125x}{100}$ will be the raised price.

In the first case 20 shillings will buy $\frac{20}{x}\times 20$ eggs, and in the second case 20 shillings will buy $20\times\frac{100}{125x}\times 20$ eggs.

Hence $$\frac{400}{x}-\frac{40000}{125x}=40,\ \text{whence } x=2.$$

5. Let $x=$ the number of sovereigns, then there are $3x$ half sovereigns and $(50-x-3x)$ shillings.

Hence the worth of the whole in shillings is $20x+30x+50-4x$, so that $20x+30x+50-4x=280$.

6. A can do $\frac{1}{20}$th of the whole in a day, and B can do $\frac{1}{12}$th. Let x be the number of days that A worked, then $14-x$ will be the number of days B worked.

Hence $$\frac{x}{20}+\frac{14-x}{12}=1.$$

7. Let x = the number bought at two a penny, then $4x$ are bought at $5d.$ a dozen, and $5x$ at $8d.$ a score. Hence whole cost in pence

$= \frac{x}{2} + 4x \times \frac{5}{12} + 5x \times \frac{8}{20}$. Also they were sold for $10x \times \frac{44}{100}$ pence. Hence

$$\frac{44x}{10} - 42 = \frac{x}{2} + \frac{5x}{3} + 2x.$$

8. Let x be the number of sovereigns, then $100 - x$ will be the number of half-crowns.

Hence the total values of the coins in shillings will be $20x + \frac{5}{2}(100 - x)$;

we therefore have $20x + \frac{5}{2}(100 - x) = 1265$.

9. Let x be the number of miles he walked. The coach took $\frac{x}{12}$ hours in going x miles, and the man took $\frac{x}{4}$ hours.

Hence, as the man had one hour's start, $\frac{x}{4} = 1 + \frac{x}{12}$, whence $x = 6$. The man rode 12×2 miles on the coach, and therefore the whole distance from A to B is $6 + 24 = 30$ miles.

10. Let x be the number of pounds allowed free of charge to each person. Then the charge on $600 - 2x$ is $3s.\ 4d. + 11s.\ 8d. = 15s.$ Also the charge on $600 - x$ would be $20s.$ Hence $\frac{600 - 2x}{600 - x} = \frac{15}{20}$, whence $x = 120$.

11. Let x, y, z be the number of days in which A, B, C respectively would separately do the work; then they would do $\frac{1}{x}$th, $\frac{1}{y}$th, $\frac{1}{z}$th of the whole respectively in a day. Hence we have

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{4}, \quad \frac{1}{x} + \frac{1}{z} = \frac{1}{6} \quad \text{and} \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{12};$$

$$\therefore \quad 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{1}{4} + \frac{1}{6} + \frac{1}{12}; \quad \therefore \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{4}.$$

Hence when they work together they do one-fourth of the whole in a day: they would therefore finish the whole in four days.

12. Let x = number of years in the father's age at present = the sum of the numbers of years in the ages of the three children. After 9 years the father will be $x + 9$, and the combined ages of the sons will be $x + 27$. Hence $(x + 27) - (x + 9)$ will be the age of the youngest son at the end of nine years; the present age of the youngest son is therefore 9. Again, after 12 years the father's age will be $x + 12$, and the combined ages of the sons will be $x + 36$; hence $x + 36 - (x + 12)$ will be the age of the middle child in 12 years, so that the present age of the middle child is 12. After 15 years the father's age will be $x + 15$, and the combined ages of the children will be $x + 45$; hence $x + 45 - (x + 15)$ will be the age of the eldest child in 15 years, so that the present age is 15. Hence $x = 9 + 12 + 15 = 36$.

13. Let x = miles per hour A travels, then $x-2$ = miles per hour B travels; $\therefore\ 3x+3(x-2)$ = distance in miles. This distance also $=4\cdot\frac{2}{3}x+4(x-3)$;

$$\therefore\ 3x+3(x-2)=\frac{8}{3}x+4(x-3).$$

14. Let x = miles per hour the man walks, and y = hours the journey takes. Then xy = distance in miles; this distance is also equal to

$$\left(x+\frac{1}{2}\right)\frac{4}{5}y \text{ and } \left(x-\frac{1}{2}\right)\left(y+2\frac{1}{2}\right).$$

Hence $$xy=\frac{4}{5}\left(x+\frac{1}{2}\right)y=\left(x-\frac{1}{2}\right)\left(y+2\frac{1}{2}\right).$$

Whence $x=2$, $y=7\frac{1}{2}$, and therefore the required distance is 15 miles. [The values $y=0$, $x=\frac{1}{2}$ satisfy the *equations*, but are not applicable to the problem.]

15. Let x and y be two of the parts; then $243-x-y$ will be the other, and we shall have $\frac{x}{2}=\frac{y}{3}=\frac{243-x-y}{4}$; whence $x=54$ and $y=81$.

16. Let there be x pounds and y shillings, then we have

$$20x+y=18\left(x+\frac{y}{12}\right), \text{ whence } y=4x;$$

and when $y=4x$ it follows that $2(100x+20y)=15(20x+y)$.

17. Let $x=A$'s share in pounds, and $y=C$'s share in pounds; then $\frac{x}{2}=B$'s share, and $y-\frac{x}{3}=D$'s share.

Hence $100+\frac{x}{2}=y+\left(y-\frac{x}{3}\right)$, and $x+\frac{x}{2}+y+y-\frac{x}{3}=1000$.

18. Let x and $\frac{3}{5}x$ be the numbers; then $x^2-\left(\frac{3}{5}x\right)^2=16$, and therefore $x=\pm 5$.

19. Let x and y be the digits; then the numbers will be $10x+y$ and $10y+x$ respectively. Hence we have

$$10x+y+10y+x=(x+y)^2, \text{ and } 10x+y-(10y+x)=5y^2.$$

From the first equation $x+y=11$ [$x+y=0$ being clearly inapplicable]; then the second equation gives $5y^2+18y-99=0$, whence $y=3$ $\left(y=-\frac{33}{5}\right.$ being clearly inapplicable$\left.\right)$.

Hence $y=3$ and then $x=8$, so that the numbers are 83 and 38.

20. Let $3x$ = the number of miles in the journey. Then

$$\frac{x}{10}+\frac{x}{9}+\frac{x}{8}=\text{number of hours he took;}$$

and this is also $$=\frac{3x}{20}+\frac{3x}{16}-\frac{1}{120}.$$

Hence $$\frac{x}{10}+\frac{x}{9}+\frac{x}{8}=\frac{3x}{20}+\frac{3x}{16}-\frac{1}{120}.$$

21. At 3 o'clock the Cambridge man has gone $54-9=45$ miles, and the Stortford man $27+9=36$. Hence they travel at 15 and 12 miles per hour respectively. We therefore have, if x be the number of hours before they met first, $15x+12x=27$. Hence $\therefore\ x=1$, and distance from Cambridge is 15 miles.

22. Let x be the number of shillings in A's share, then $x-100$ will be the number in B's share, and $x(x-100)$ will be the number in C's share.

Hence $x+x-100+x(x-100)=20300$, giving $x=200$.

23. Let x be the number of posts in a mile, then $\frac{1760}{x}$ = number of yards between two posts.

Then $$\frac{1760}{x-1}-\frac{1760}{x}=2\frac{14}{15},\ \text{whence } x=25.$$

24. Let x be the greater and y the less. Then $(x+y)x=144$ and $(x-y)y=14$. Hence $7x^2-65xy+72y^2=0$ that is $(x-8y)(7x-9y)=0$.

$$7x-9y=0 \text{ and } x(x+y)=144 \text{ give } x=\pm 9,\ y=\pm 7;$$

also $$x-8y=0 \text{ and } x(x+y)=144 \text{ give } x=\pm 8\sqrt{2},\ y=\pm\sqrt{2}.$$

The negative values are to be rejected since x is greater than y; the numbers required are therefore 9 and 7 or $8\sqrt{2}$ and $\sqrt{2}$.

25. Let $x=A$'s rate per hour in miles, and $y=B$'s rate. Then $5x+5y$ = required distance. If A were to walk 1 mile faster he would get to the meeting place in $\frac{5x}{x+1}$ hours, and by supposition B would get to this place 4 hours after A started; $\therefore\ \frac{5x}{x+1}=4$. Similarly $6=\frac{5y}{y-1}$. Hence $x=4$ and $y=6$, whence it follows that the distance is 50 miles.

26. Let x be the number of men in the side of the solid square; then x^2 is the whole number of men. But the number of men in the hollow square four deep and with $x+16$ in each outer side is $(x+16)^2-(x+8)^2=16x+192$.

Hence we have $$x^2=16x+192.$$

27. Let x be the units digit and y the tens digit. Then we have

$$10y+x=7(x+y),$$

whence $y=2x$, from which it follows that $10x+y=4(x+y)$.

28. Let A walk at the rate of x miles per hour. Then A takes $\frac{7}{x}$ hours on the journey; hence B takes $\frac{7}{x}-\frac{1}{3}$ on his journey and therefore B overtakes A in $\frac{7}{2x}-\frac{1}{6}$ hours; in which time B has walked $4\left(\frac{7}{2x}-\frac{1}{6}\right)$ miles. But at this time A has walked $x\left\{\frac{7}{2x}-\frac{1}{6}+\frac{1}{3}\right\}$ miles.

Hence $$4\left(\frac{7}{2x}-\frac{1}{6}\right)=x\left(\frac{7}{2x}+\frac{1}{6}\right).$$

29. Let x be the number of hours they rode before they met, then A takes $x+4$ and B takes $x+1$ hours altogether. Hence A's rate : B's rate $=x+1:x+4$. But B rode in x hours the same distance that A rode in 4 hours; hence A's rate : B's rate $=x:4$. Hence $x+1:x+4=x:4$, so that $x=2$ and therefore B's whole time was 3 hours.

30. Let x be the hundreds digit, y the tens and z the units digit. Then $x+y+z=10$ (i), and $y=x+z$ (ii). Also $100z+10y+x=100x+10y+z+99$, whence $z-x=1$ (iii). From (i), (ii), (iii) we have $x=2$, $y=5$ and $z=3$.

31. Let x gallons be taken from the first, and therefore $14-x$ from the second. Then $\frac{x}{4}+\frac{3}{8}(14-x)=$quantity of wine taken $=5$.

32. Let x be the number of minutes past 10 o'clock at the time in question. Then $x+6+30=50+\frac{x-3}{12}$, whence $x=15$.

33. We are told that C rides $4\frac{1}{2}$ miles in half an hour; hence C rides 9 miles an hour. Hence C rides 18 miles between 5 o'clock and 7 o'clock, and therefore B drives 18 miles between 4 and 7, so that B drives at the rate of 6 miles an hour. Again C rides $22\frac{1}{2}$ miles between 5 o'clock and 7.30, and therefore A walks $22\frac{1}{2}$ miles in $4\frac{1}{2}$ hours; thus A walks 5 miles an hour. Now let $x=$the number of hours A walks before B overtakes him; then $5x=6(x-1)$; $\therefore\ x=6$. Hence B catches A at 9 o'clock, 30 miles from Cambridge.

34. Let x and y be the number of yards the trains travel in one second, then $x-y=$the number of yards per second the fast train gains on the slow.

Hence $$\frac{60+72}{x-y}=12, \text{ and } \frac{60+72}{x-\frac{3}{2}y}=24,$$

whence $x=22$ and $y=11$. Hence the fast train goes $\frac{22\times 60\times 60}{1760}=45$ miles per hour.

35. Let x be the number of pounds A gives to each; then $\frac{180}{x}=$ the number of people to whom A gives. Hence we have

$$(x+6)\left(\frac{180}{x}-40\right)=180; \quad \therefore x=\mathbf{3}.$$

36. Let $x=$ the number of miles per hour the first goes, and $y=$ the number of hours on the journey; then $xy=$ the distance required.

This distance is also

$$\left(x-\frac{1}{2}\right)\left(y+\frac{3}{2}\right) \text{ and } \left(x-\frac{1}{2}-\frac{3}{4}\right)\left(y+\frac{3}{2}+\frac{5}{2}\right).$$

Hence $$xy=\left(x-\frac{1}{2}\right)\left(y+\frac{3}{2}\right)=\left(x-\frac{5}{4}\right)(y+4),$$

whence $x=12\frac{1}{2}$ and $y=36$, and therefore $xy=450$.

37. Let A walk at x miles per hour, and B at y miles per hour, and let d be the distance from P to Q. Then by the question (1) A walks $d-2$ miles in one hour less than B, (2) A and B together walk 4 miles in 32′ or $7\frac{1}{2}$ miles an hour, and (3) A walks $d+2$ miles while B walks $d-2$ miles. Hence from (1) we have $\frac{d-2}{x}-\frac{d-2}{y}+1=0$; from (2) we have $x+y=7\frac{1}{2}$; and from (3) we have $\frac{d+2}{x}-\frac{d-2}{y}=0$. From the first and third we have at once $x=\mathbf{4}$, and then $y=3\frac{1}{2}$ from the second; hence $d+2:d-2=8:7$, whence $d=30$.

XVI. Pages 190—199.

1. Since $$\frac{x}{y+z}=\frac{a}{1}, \quad \frac{x}{x+y+z}=\frac{a}{a+1};$$

so also $$\frac{y}{x+y+z}=\frac{b}{b+1} \text{ and } \frac{z}{x+y+z}=\frac{c}{c+1}.$$

Hence, by addition,

$$1=\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}.$$

2. From the first equation $x = -\frac{b}{a}y$. Substitute in the second equation; then $\left(\frac{cb^2}{a^2} - \frac{db}{a} + e\right)y^2 = 0$; $\therefore\ cb^2 + ea^2 - abd = 0$, or else $y = 0$ and therefore also $x = 0$.

3. Since $$\frac{y-z}{y+z} = \frac{a}{1},\quad \frac{2y}{2z} = \frac{1+a}{1-a}.$$

So also $$\frac{2z}{2x} = \frac{1+b}{1-b} \text{ and } \frac{2x}{2y} = \frac{1+c}{1-c}.$$

Hence, by multiplication,

$$1 = \frac{(1+a)(1+b)(1+c)}{(1-a)(1-b)(1-c)},$$

which is equivalent to $a+b+c+abc = 0$.

4. Subtract the first from the sum of the second and third; then

$$2\frac{z}{y} = b + c - a.$$

Similarly $$2\frac{x}{z} = c + a - b \text{ and } 2\frac{y}{x} = a + b - c.$$

Hence, by multiplication,

$$8 = (b+c-a)(c+a-b)(a+b-c).$$

5. $$z = \frac{1}{1-y} \text{ and } \frac{1}{x} = \frac{1}{y-1};$$

$$\therefore\ z + \frac{1}{x} = \frac{1}{1-y} + \frac{y}{y-1} = 1.$$

6. We have $$\frac{2a}{b+d} = \frac{1}{x} - x,$$

and $$\frac{2c}{b-d} = \frac{1}{x} + x.$$

Square and subtract; then $$\frac{4c^2}{(b-d)^2} - \frac{4a^2}{(b+d)^2} = 4.$$

7. Multiply the first three equations by x, y, z respectively and add; then we have

$$x^3 + y^3 + z^3 - 3xyz = ax + by + cz = d.$$

Also $$(x^2 - yz)^2 - (y^2 - zx)(z^2 - xy) = a^2 - bc;$$

$$\therefore\ x(x^3 + y^3 + z^3 - 3xyz) = a^2 - bc.$$

Hence $$dx = a^2 - bc,$$

and similarly $$dy = b^2 - ca \text{ and } dz = c^2 - ab.$$

Hence $$a^3 + b^3 + c^3 - 3abc = dax + dby + dcz = d^2.$$

8. From (i) and (ii) we have $a^2 - b^2 = 2\Sigma yz$. From (iii)

$$c^3 = (x+y+z)(x^2+y^2+z^2-yz-zx-xy) = a\left\{b^2 - \frac{1}{2}(a^2-b^2)\right\}.$$

Hence $$2c^3 = 3ab^2 - a^3.$$

9. Put $$bz+cy = cx+az = ay+bx = \lambda;$$

then $$\frac{y}{b}+\frac{z}{c} = \frac{\lambda}{bc}, \quad \frac{z}{c}+\frac{x}{a} = \frac{\lambda}{ca}$$

and $$\frac{x}{a}+\frac{y}{b} = \frac{\lambda}{ab}.$$

Hence $$\frac{2x}{a} = \frac{\lambda}{abc}(b+c-a), \text{ \&c.}$$

Substitute in the given relation $\Sigma x^2 - 2\Sigma yz = 0$ and we have

$$\Sigma a^2(b+c-a)^2 - 2\Sigma bc(c+a-b)(a+b-c) = 0,$$

which will be found to reduce to $\Sigma a^4 - 2\Sigma b^2c^2 = 0$, that is

$$-(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 0.$$

10. We have $$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2 = 1,$$

that is $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{2xyz}{abc}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right) = 1;$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

11. We have

$$yz(x-y) = y-z, \; zx(y-z) = z-x \text{ and } xy(z-x) = x-y.$$

Hence $$x^2y^2z^2(x-y)(y-z)(z-x) = (y-z)(z-x)(x-y);$$

$\therefore$ either $$x^2y^2z^2 = 1 \text{ or } (y-z)(z-x)(x-y) = 0.$$

12. Multiply the corresponding members of the second and third equations; then

$$yz = a^2yz + x(bcx+cay+abz);$$

$$\therefore \frac{x^2}{1-a^2} = \frac{xyz}{bcx+cay+abz} = \text{similarly } \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2}.$$

13. We have $(x^2-y^2-z^2)^2-4y^2z^2=4(a^2-1)y^2z^2$;

$$\therefore \frac{a^2-1}{x^2}=\frac{x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2}{4x^2y^2z^2}$$

$$=\frac{b^2-1}{y^2}=\frac{c^2-1}{z^2}.$$

14. From the first two equations

$$y=\{a(c+dx)+b(a+bx)\}/\{c(c+dx)+d(a+bx)\},$$

and from the third equation $y=(a-cx)/(dx-b)$.

Equating the values of y, we have after reduction

$$(ad+b^2+bc+c^2)\{dx^2+(c-b)x-a\}=0.$$

Hence either $ad+b^2+bc+c^2=0$ .....(i)

or else $dx^2+(c-b)x-a=0.$

Similarly unless (i) is true we have

$$dy^2+(c-b)y-a=0 \text{ and } dz^2+(c-b)z-a=0;$$

Hence (i) is true or else, from Art. 127, $d=c-b=a=0$.

15. By continued multiplication, we have

$$1+1+\frac{y^2z^2}{x^4}+\frac{z^2x^2}{y^4}+\frac{x^2y^2}{z^4}+\frac{x^4}{y^2z^2}+\frac{y^4}{z^2x^2}+\frac{z^4}{x^2y^2}=lmn;$$

$$\therefore 2+(l^2-2)+(m^2-2)+(n^2-2)=lmn,$$

or $$l^2+m^2+n^2-lmn=4.$$

16. From the first three equations we have

$$bx+\frac{c}{x}=-l,\ cy+\frac{a}{y}=-m,\ az+\frac{b}{z}=-n.$$

By continued multiplication, since $xyz=1$, we have

$$2abc+a\left(\frac{c^2}{x^2}+b^2x^2\right)+b\left(c^2y^2+\frac{a^2}{y^2}\right)+c\left(a^2z^2+\frac{b^2}{z^2}\right)=-lmn;$$

$$\therefore 2abc+a(l^2-2bc)+b(m^2-2ca)+c(n^2-2ab)=-lmn.$$

17. We have $\frac{y}{z}+\frac{z}{y}=a,\ \frac{z}{x}+\frac{x}{z}=b,\ \frac{x}{y}+\frac{y}{x}=c.$

Hence

$$abc=2+\frac{z^2}{y^2}+\frac{y^2}{z^2}+\frac{x^2}{z^2}+\frac{z^2}{x^2}+\frac{y^2}{x^2}+\frac{x^2}{y^2}=2+a^2-2+b^2-2+c^2-2.$$

18. (i) By continued multiplication

$$2abc + a\left(\frac{c^2z^2}{y^2} + \frac{b^2y^2}{z^2}\right) + b\left(\frac{a^2x^2}{z^2} + \frac{c^2z^2}{x^2}\right) + c\left(\frac{b^2y^2}{x^2} + \frac{a^2x^2}{y^2}\right) = abc;$$

$$\therefore\ 2abc + a(a^2 - 2bc) + b(b^2 - 2ca) + c(c^2 - 2ab) = abc.$$

(ii) From the first two equations we have

$$\frac{a}{\frac{y}{x} + \frac{z}{y}} = \frac{b}{\frac{x}{y} + \frac{z}{x}} = \frac{c}{1 - \frac{xy}{z^2}}.$$

Substituting in the third equation, we have

$$\frac{x}{y}\left(\frac{y}{x} + \frac{z}{y}\right) + \frac{y}{x}\left(\frac{x}{y} + \frac{z}{x}\right) = 1 - \frac{xy}{z^2}.$$

19. From the first three equations, we have by continued multiplication

$$abcxyz + x^2y^2z^2 + bcy^2z^2 + caz^2x^2 + abx^2y^2 + xyz\,(ax^2 + by^2 + cz^2) = a^2b^2c^2,$$

or
$$a^2b^2c^2 + \Sigma bcy^2z^2 + abc\Sigma ax^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Also
$$a^2x^2 + y^2z^2 + 2axyz = b^2c^2;$$

$$\therefore\ a^2bcx^2 + bcy^2z^2 = b^3c^3 - 2a^2b^2c^2.$$

So also
$$b^2cay^2 + caz^2x^2 = c^3a^3 - 2a^2b^2c^2,$$

and
$$c^2abz^2 + abx^2y^2 = a^3b^3 - 2a^2b^2c^2.$$

Add; then from (i) we have

$$-a^2b^2c^2 = \Sigma b^3c^3 - 6a^2b^2c^2.$$

20. Put λ for each of the equal fractions; then $x(x - y - z) = a\lambda$, &c

Hence
$$\lambda(a - b - c) = (x - y + z)(x + y - z), \text{ \&c.}$$

$$\therefore\ -x + y + z = \pm\frac{\sqrt{\{(a - b - c)(-a + b - c)(-a - b + c)\}}}{a - b - c}, \text{ \&c.};$$

$$\therefore\ \frac{x}{a(a - b - c)} = \frac{y}{b(b - c - a)} = \frac{z}{c(c - a - b)}.$$

Substituting in $ax + by + cz = 0$, we have the required relation, namely

$$\Sigma a^2(a - b - c) = 0.$$

21. By multiplication

$$a^2b^2c^2x^2y^2z^2 = \alpha^2\beta^2\gamma^2(y + z)^2(z + x)^2(x + y)^2;$$

$$\therefore \pm\frac{abc}{\alpha\beta\gamma} = (y + z)(z + x)(x + y)/xyz.$$

Also
$$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} - 4 = \frac{x(y + z)^2 + y(z + x)^2 + z(x + y)^2 - 4xyz}{xyz}$$

$$= \frac{(y+z)(z+x)(x+y)}{xyz} = \pm\frac{abc}{\alpha\beta\gamma}.$$

22. $2b^2c^2 - a^4 = 2x^4 - y^4 - z^4 + 2(x^3y + x^3z - y^3z - z^3y)$
$+ (2z^2x^2 + 2x^2y^2 - y^2z^2) + 2xyz(x+y+z).$

Hence

$$2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 = 3\Sigma y^2z^2 + 6xyz\Sigma x = 3(\Sigma yz)^2 = 0.$$

But the factors of $2\Sigma b^2c^2 - \Sigma a^4$ are known to be $a \pm b \pm c$.

23. If $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{a+b+c} = 0;$$

then $$(bc + ca + ab)(a+b+c) - abc = 0,$$

that is $$(b+c)(c+a)(a+b) = 0.$$

Hence either $b+c=0$ or $c+a=0$ or $a+b=0$.

Now, if $b+c=0$, then $b^{2n+1} + c^{2n+1} = 0$,

and therefore $$\frac{1}{a^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} - \frac{1}{(a+b+c)^{2n+1}} = 0.$$

Similarly if $c+a=0$ or $a+b=0$.

24. If $$\frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab} = 1,$$

then $$a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2) - 2abc = 0,$$

that is $$(b+c-a)(c+a-b)(a+b-c) = 0.$$

Now, if $b+c-a=0$, the second expression will vanish provided

$$(-1)^{2n+1} + (1)^{2n+1} + (1)^{2n+1} = 1,$$

which is obvious.

25. We have $$\Sigma a^2x^3 = 0, \quad \Sigma a^2x^3 = 0,$$

and $$\frac{1}{x} - a^2 = \frac{1}{y} - b^2 = \frac{1}{z} - c^2.$$

Put $$\frac{1}{x} - a^2 = \lambda;$$ then $a^2x = 1 - \lambda x$, &c.

Hence, $$\Sigma a^4x^3 = \Sigma a^2x^2(1 - \lambda x) = \Sigma a^2x^2 - \lambda\Sigma a^2x^3 = 0.$$

Also $\Sigma a^6x^3 = \Sigma a^4x^2(1-\lambda x) = \Sigma a^4x^2 - \lambda\Sigma a^4x^3 = \Sigma a^4x^2.$

26. Put λ for each of the equal expressions; then we have

$$x^3 - ayz = \lambda x^2,\ y^3 - azx = \lambda y^2 \text{ and } z^3 - axy = \lambda z^2.$$

Hence $(x-y)(x^2+xy+y^2+az) = \lambda(x+y)(x-y).$

$\therefore\ x^2+xy+y^2+az = \lambda(x+y)$, since $x \neq y$.

Similarly $x^2+xz+z^2+ay = \lambda(x+z).$

Hence $(y-z)(x+y+z-a) = \lambda(y-z)$;

$\therefore\ x+y+z-a = \lambda$, since $y \neq z$.

27. From the given equations we have by subtraction

$$3(z-y)yz = z(z-x)^2 - y(x-y)^2.$$

Hence, as $y \neq z$, we have

$$3yz = z^2+yz+y^2-2x(z+y)+x^2,$$

that is $$x^2+y^2+z^2-2yz-2zx-2xy = 0.$$

Hence $$2a-3y = \frac{(z-x)^2}{y} = 2z+2x-y,$$

whence $$x+y+z = a.$$

Then $$\frac{(y-z)^2}{x} = 2y+2z-x = 2a-3x.$$

28. We have $$x + \frac{yz-x^2}{x^2+y^2+z^2} = y + \frac{xz-y^2}{x^2+y^2+z^2}.$$

$$\therefore\ (x-y)(x^2+y^2+z^2-x-y-z) = 0.$$

Hence $$z + \frac{xy-z^2}{x^2+y^2+z^2} = z + \frac{xy-z^2}{x+y+z} = \frac{yz+zx+xy}{x+y+z}$$

$$= x + \frac{yz-x^2}{x+y+z} = x + \frac{yz-x^2}{x^2+y^2+z^2}.$$

The expressions will all vanish provided $2\Sigma yz = 0$, and this will clearly be the case when $$x^2+y^2+z^2 = x+y+z = 1.$$

29. From $$y^3+z^3+m(y+z) = z^3+x^3+m(z+x),$$

we have $$(y-x)(x^2+xy+y^2+m) = 0;$$

$$\therefore\ x^2+xy+y^2+m = 0 \text{ and similarly } y^2+yz+z^2+m = 0.$$

Hence $$(x-z)(x+y+z) = 0 \text{ and therefore } x+y+z = 0.$$

Then $$y^3+z^3+m(y+z) = (y+z)\{y^2-yz+z^2+m\} = (-x)(-2yz).$$

30. From $y^2+z^2+myz=z^2+x^2+mzx,$

we have $(y-x)(x+y+mz)=0.$

$\therefore\ x+y+mz=0$ and similarly $y+z+mx=0;$

$\therefore\ (x-z)(1-m)=0;\ \therefore\ m=1.$

Then, since $m=1$, we have $x+y+z=0.$

Also $y^2+z^2+myz=y^2+z^2+yz$

$$=\frac{1}{2}\{y^2+z^2+(y+z)^2\}=\frac{1}{2}(x^2+y^2+z^2).$$

31. We have $\dfrac{\{3x-(x+y+z)\}^3}{x}=\dfrac{\{3y-(x+y+z)\}^3}{y};$

$$\therefore\ 27x^2-27x(x+y+z)+9(x+y+z)^2-\frac{(x+y+z)^3}{x}$$

$$=27y^2-27y(x+y+z)+9(x+y+z)^2-\frac{(x+y+z)^3}{y}.$$

Whence $27xyz-(x+y+z)^3=0$, and the symmetry of this result shews that each of the given expressions $=\dfrac{(2z-x-y)^3}{z}.$

32. We have $(ac-bd)^2+(ad-bc)^2=0;$

$\therefore\ ac=bd$ and $ad=bc;$

$\therefore\ a^2cd=b^2cd,$

whence $a=\pm b$ and then $c=\pm d.$

33. We have $(ax-b)^2+(bx-c)^2=0;$

$\therefore\ ax-b=0$ and $bx-c=0;$

$$\therefore\ \frac{b}{a}=\frac{c}{b}=x.$$

34. We have $(xb-ya)^2+(xc-za)^2+(yc-zb)^2=0.$

Hence $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

35. (iv). We have $(n-1)\Sigma a^2-2\Sigma ab=0$, that is $\Sigma(a-b)^2=0$, every letter being taken with every other. Hence all the letters are equal.

36. $a^4+b^4+c^4+d^4-4abcd=(a^2-b^2)^2+(c^2-d^2)^2+2(ab-cd)^2.$

Hence we have $a^2-b^2=c^2-d^2=ab-cd=0$, from which it follows that $a=b=c=d$ provided the quantities are all of same sign.

37. We have

$$x^2+2x(a_1-a_2)+(a_1-a_2)^2$$
$$+x^2+2x(a_2-a_3)+(a_2-a_3)^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+x^2+2x(a_{n-1}-a_n)+(a_{n-1}-a_n)^2=0,$$

that is $$(x+a_1-a_2)^2+(x+a_2-a_3)^2+\ldots+(x+a_{n-1}-a_n)^2=0.$$

Hence $$x+a_1-a_2=x+a_2-a_3=\ldots=x+a_{n-1}-a_n=0.$$

38. The expressions on the two sides of the sign of equality when arranged according to powers of a are easily seen to be

$$a^3(b+c)+a^2bc+a(b+c)(b^2+c^2)+bc(b^2+c^2).$$

39. The expression on the left would vanish if $b=c$, and therefore $b-c$ is a factor. So also $c-a$, $a-b$, $a-d$, $b-d$ and $c-d$ are factors. Hence, as the expression is of the *sixth* degree, it must

$$=L(b-c)(c-a)(a-b)(a-d)(b-d)(c-d),$$

when L is numerical.

Let $a=0$, $b=1$, $c=2$, $d=-1$; then we have

$$4^4\,.\,-1\,.\,1+2^4\,.\,2\,.\,2=L\,.\,-1\,.\,2\,.\,-1\,.\,1\,.\,2\,.\,3,$$

whence $L=-16$.

40. $2a+b+c$ is a factor of $(2a+2b+2c)^3-(b+c)^3$ and also of

$$(c+a)^3+(a+b)^3.$$

Hence $2a+b+c$ is a factor of the expression on the left, and similarly $a+2b+c$ and $a+b+2c$ are factors. Hence the expression on the left

$$=L(2a+b+c)(2b+c+a)(2c+a+b).$$

Comparing the coefficients of a^3 we have $2L=6$.

41. Put $a=0$ in the expression on the left, and the result vanishes identically. Hence a is a factor, and similarly b, c, d are factors. Hence, as the expression is symmetrical and of the *fifth* degree, it must be equal to

$$Labcd(a+b+c+d).$$

Put $a=b=c=d=1$; then $4^5-4\,.\,3^5+6\,.\,2^5-4\,.\,1=4L$; $\therefore\ L=60$.

42. $$abc\{(a+b+c)^3-a^3-b^3-c^3\}=3abc(b+c)(c+a)(a+b);$$

similarly $$(bc+ca+ab)^3-b^3c^3-c^3a^3-a^3b^3$$
$$=3(ca+ab)(ab+bc)(bc+ca)=3abc(b+c)(c+a)(a+b).$$

Hence $$abc(a+b+c)^3-(bc+ca+ab)^3$$
$$=abc(a^3+b^3+c^3)-(b^3c^3+c^3a^3+a^3b^3).$$

43. Since $A^3 - 3AB^2 + 2B^3 = (A-B)^2(A+2B)$, the expression on the left

$$= (a^2+b^2+c^2-bc-ca-ab)^2\{a^2+b^2+c^2+2(bc+ca+ab)\}$$
$$= (a^2+b^2+c^2-bc-ca-ab)^2(a+b+c)^2$$
$$= (a^3+b^3+c^3-3abc)^2.$$

44. Left side

$$= abc(a+b+c) + b^2c^2+c^2a^2+a^2b^2 - bc(b^2+c^2) - ca(c^2+a^2) - ab(a^2+b^2)$$
$$= 2abc(a+b+c) + b^2c^2+c^2a^2+a^2b^2 - (bc+ca+ab)(a^2+b^2+c^2)$$
$$= (bc+ca+ab)^2 - (bc+ca+ab)(a^2+b^2+c^2)$$
$$= \Sigma bc \,.\, (\Sigma bc - \Sigma a^2).$$

45. First line

$$= 2a^4 - b^4 - c^4 + 2(a^3c + a^3b - b^3c - bc^3) + 2a^2b^2 + 2a^2c^2 - b^2c^2 + 2abc(a+b+c).$$

Hence sum $= 3(b^2c^2+c^2a^2+a^2b^2) + 6abc(a+b+c)$,
the terms of the form a^4 and a^3b all vanishing.

Hence sum $= 3(bc+ca+ab)^2$.

46. Expression on the left

$$= \frac{1}{2}[(3a-b-c)+(3b-c-a)+(3c-a-b)]$$
$$\times [\{(3b-c-a)-(3c-a-b)\}^2 + \text{two like terms}]$$
$$= \frac{1}{2}(a+b+c)[(4b-4c)^2 + \&c.]$$
$$= 16(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
$$= 16(a^3+b^3+c^3-3abc).$$

47. Expression on the left

$$= \frac{1}{2}[(na-b-c)+\ldots][\{(nb-c-a)-(nc-a-b)\}^2+\ldots]$$
$$= \frac{1}{2}(n-2)(n+1)^2(a+b+c)[(b-c)^2+\ldots]$$
$$= (n-2)(n+1)^2(a^3+b^3+c^3-3abc).$$

48. $(x^2+2yz)+(y^2+2zx)+(z^2+2xy) = (x+y+z)^2$,

$(x^2+2yz)+\omega(y^2+2zx)+\omega^2(z^2+2xy) = (x+\omega^2y+\omega z)^2$,

and $x^2+2yz+\omega^2(y^2+2zx)+\omega(z^2+2xy) = (x+\omega y+\omega^2 z)^2$.

Hence

$$(x^2+2yz)^3 + (y^2+2zx)^3 + (z^2+2xy)^3 - 3(x^2+2yz)(y^2+2zx)(z^2+2xy)$$
$$= (x^3+y^3+z^3-3xyz)^2.$$

49. The expression on the left is the continued product of

$$(by+az)+(bz+ax)+(bx+ay),$$

$$(by+az)+\omega\,(bz+ax)+\omega^2\,(bx+ay),$$

and $$(by+az)+\omega^2\,(bz+ax)+\omega\,(bx+ay).$$

That is, of $$(a+b)\,(x+y+z),$$

$$(\omega a+\omega^2 b)\,(x+\omega y+\omega^2 z) \text{ and } (\omega^2 a+\omega b)\,(x+\omega^2 y+\omega z).$$

But $$(a+b)\,(\omega a+\omega^2 b)\,(\omega^2 a+\omega b)=a^3+b^3,$$

and $$(x+y+z)\,(x+\omega y+\omega^2 z)\,(x+\omega^2 y+\omega z)=x^3+y^3+z^3-3xyz.$$

50. Since $X^3+Y^3=(X+Y)\,(X+\omega Y)\,(X+\omega^2 Y)$, the expression on the left is the continued product of

$$2\,(b-c)\,(x-a)+(\omega+\omega^2)\,(c-a)\,(x-b)+(\omega^2+\omega)\,(a-b)\,(x-c)$$
$$=2\,(b-c)\,(x-a)-(c-a)\,(x-b)-(a-b)\,(x-c)=3\,(b-c)\,(x-a),$$

$$(b-c)\,(x-a)\,(1+\omega)+(\omega+1)\,(c-a)\,(x-b)+2\omega^2\,(a-b)\,(x-c)$$
$$=3\omega^2\,(a-b)\,(x-c),$$

and $$(b-c)\,(x-a)\,(1+\omega^2)+2\omega\,(c-a)\,(x-b)+(\omega^2+1)\,(a-b)\,(x-c)$$
$$=3\omega\,(c-a)\,(x-b).$$

Hence the expression on the left

$$=27\,(b-c)\,(c-a)\,(a-b)\,(x-a)\,(x-b)\,(x-c).$$

51. $$(a^2+b^2)\,(c^2+d^2)=(ac\pm bd)^2+(ad\mp bc)^2.$$

Hence the product of any two expressions of the form x^2+y^2 is of the same form; the same will therefore be true when a third expression is taken; and so on.

52. The coefficients of a^2, b^2, c^2 and d^2 on the right are each seen to be $p^2+q^2+r^2+s^2$; also the coefficients of ab and of the terms of like form are easily seen to be zero. Hence the expression on the right

$$=(a^2+b^2+c^2+d^2)\,(p^2+q^2+r^2+s^2).$$

Thus the product of any two expressions which are the sum of four squares can be expressed as the sum of four squares; the same will therefore be true when a third expression is taken, &c.

53. $$(x^2+xy+y^2)\,(a^2+ab+b^2)=(x-\omega y)\,(x-\omega^2 y)\,(a-\omega b)\,(a-\omega^2 b).$$

Now $$(x-\omega y)\,(a-\omega b)=ax-\omega\,(ay+bx)+\omega^2 by$$
$$=ax-by-\omega\,(ay+bx+by),$$

since $$1+\omega+\omega^2=0.$$

So also $$(x-\omega^2 y)\,(a-\omega^2 b)=ax-by-\omega^2\,(ay+bx+by).$$

And $$\{ax-by-\omega\,(ay+bx+by)\}\,\{ax-by-\omega^2\,(ay+bx+by)\}$$
$$=(ax-by)^2+(ax-by)\,(ay+bx+by)+(ay+bx+by)^2$$

which is of the form X^2+XY+Y^2.

54. Let $x^2+pxy+qy^2=(x+\alpha y)(x+\beta y)$;

then $a^2+pab+qb^2=(a+\alpha b)(a+\beta b)$,

and $\alpha\beta=q, \quad \alpha+\beta=-p.$

Hence $(x^2+pxy+qy^2)(a^2+pab+qb^2)$
$=(x+\alpha y)(x+\beta y)(a+\alpha b)(a+\beta b).$

Now $(x+\alpha y)(a+\beta b)=ax+\alpha ay+(-p-\alpha)bx+qby$
$=(ax-pbx+qby)+\alpha(ay-bx);$

so also $(x+\beta y)(a+\alpha b)=(ax-pbx+qby)+\beta(ay-bx).$

Hence $(x^2+pxy+qy^2)(a^2+pab+qb^2)$
$=(ax-pbx+qby)^2+p(ax-pbx+qby)(ay-bx)+q(ay-bx)^2.$

55. (i) We have

$2s^3-s^2(a+b+c)+s\{2bc+2ca+2ab-a(b+c)-b(c+a)-c(a+b)\}$
$+3abc-2abc=abc.$

(ii) Multiply by 8 the expression on the left; then we have

$(-a+b+c)^3+(a-b+c)^3+(a+b-c)^3+24abc$
$=\Sigma a^3+3\Sigma a^2b+6abc=(a+b+c)^3.$

(iii) $(b+c)s(s-a)+a(s-b)(s-c)-2sbc$
$=s^2(a+b+c)-2s(bc+ca+ab)+abc,$

and the symmetry of this result shews that the three expressions are all equal.

(iv) $\Sigma a(b-c)(s-a)^2=s^2\Sigma a(b-c)-2s\Sigma a^2(b-c)+\Sigma a^3(b-c).$

Now $\Sigma a(b-c)=0, \quad \Sigma a^2(b-c)=-(b-c)(c-a)(a-b),$

and $\Sigma a^3(b-c)=-(b-c)(c-a)(a-b)(a+b+c)=2s\Sigma a^2(b-c).$

Hence $\Sigma a(b-c)(s-a)^2=0.$

(v) $s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)-(s-a)(s-b)(s-c)$
$=2s^3-s^2(a+b+c)+abc=abc.$

(vi) Squaring (v) we have

$s^2(s-b)^2(s-c)^2+s^2(s-c)^2(s-a)^2+s^2(s-a)^2(s-b)^2+(s-a)^2(s-b)^2(s-c)^2$
$+2s(s-a)(s-b)(s-c)\{s(s-a)+s(s-b)+s(s-c)-(s-b)(s-c)$
$-(s-c)(s-a)-(s-a)(s-b)\}=a^2b^2c^2$

But $2\{\Sigma s(s-a)-\Sigma(s-b)(s-c)\}=2s(a+b+c)-2bc-2ca-2ab$
$=a^2+b^2+c^2.$

This proves the theorem.

56. $4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2$
$=(2bc+2ad-b^2-c^2+a^2+d^2)(2bc+2ad+b^2+c^2-a^2-d^2)=\{(a+d)^2-(b-c)^2\}$
$\{(b+c)^2-(a-d)^2\}=(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)$
$=16(s-a)(s-b)(s-c)(s-d).$

In the second expression :—

The coefficient of s^4 on the left $=2$,

the coefficient of s^3
$$=a+b+c+d-2(a+b+c+d)=-(a+b+c+d),$$

the coefficient of s^2
$$=-a(b+c+d)-b(c+d+a)-c(d+a+b)-d(a+b+c)+2\Sigma ab=0,$$

the coefficient of s
$$=3\Sigma abc-2\Sigma abc-\Sigma abc=0,$$

and the coefficient of s^0
$$=-4abcd+2abcd=-2abcd.$$

Hence the whole expression on the left
$$=2s^4-(a+b+c+d)s^3-2abcd=-2abcd.$$

57. The expression on the left
$=ad(b+c)^2+bc(a-d)^2+ab(c+d)^2+cd(a-b)^2+ac(b+d)^2+bd(a-c)^2$
$+4abcd=ad(b^2+c^2)+bc(a^2+d^2)+ab(c^2+d^2)+cd(a^2+b^2)+ac(b^2+d^2)$
$+bd(a^2+c^2)+4abcd$
$=bcd(b+c+d)+cda(c+d+a)+dab(d+a+b)+abc(a+b+c)+4abcd$
$=-abcd-abcd-abcd-abcd+4abcd=0.$

58. Arrange according to powers of d; then we have
$$b^2d^2-2abcd+a^2c^2=0, \text{ whence } ac=bd.$$

59. First substitute $-(a+b)$ for c in the expression on the left, and then put $a=0$ in the result; we then have
$$4b^3(y-z)^3-6b^3(y-z)(x^2+y^2+z^2)-4b^3(y-z)(z-x)(x-y)$$
$$=2b^3(z-y)(x+y+z)^2=0.$$

Hence a is a factor. Similarly it can be shewn that b and $a+b$ are factors.

Hence abc must be a factor of the original expression. So also xyz must be a factor.

Hence the expression $=Labcxyz$.

Now put $a=b=x=y=1$ and $c=z=-2$; then $4.6^3-3.6.6.6=L.4$, whence $L=54$.

60. Since $a+b+c=0$, we may consider a, b, c to be the roots of the cubic $x^3+px+q=0$. We then have, as in Art. 129,

$\Sigma a^2=-2p$, $\Sigma a^3=-3q$, $\Sigma a^4=2p^2$, $\Sigma a^5=5pq$, $\Sigma a^6=3q^2-2p^3$ and $\Sigma a^7=-7p^2q$.

Hence (i) $\frac{1}{7}\Sigma a^7 = -p^2q = -q \cdot \frac{1}{2}\Sigma a^4 = \frac{1}{2}abc \cdot \Sigma a^4$.

(ii) $\frac{1}{7}\Sigma a^7 = -p^2q = \frac{1}{3}\Sigma a^3 \cdot \frac{1}{2}\Sigma a^4$.

(iii) $\Sigma a^6 = 3q^2 - 2p^3 = 3a^2b^2c^2 + \frac{1}{4}(a^2+b^2+c^2)^3$.

(iv) $\frac{1}{7}\Sigma a^7 \cdot \frac{1}{3}\Sigma a^3 = p^2q^2 = \frac{1}{25}(\Sigma a^5)^2$.

61. $(a^3+b^3+c^3+d^3)^2 = \{a^3+b^3+c^3-(a+b+c)^3\}^2 = 9(b+c)^2(c+a)^2(a+b)^2$.

Also $9(bcd+cda+dab+abc)^2$

$= 9\{abc-(bc+ca+ab)(a+b+c)\}^2 = 9(b+c)^2(c+a)^2(a+b)^2$.

And $9(bc-ad)(ca-bd)(ab-cd)$

$= 9\{bc+a(a+b+c)\}\{ca+b(a+b+c)\}\{ab+c(a+b+c)\}$

$= 9(a+b)(a+c)(b+c)(b+a)(c+a)(c+b)$.

Hence each of the three expressions

$= 9(b+c)^2(c+a)^2(a+b)^2$.

62. We have to shew that

$\{bc(b-c)+ca(c-a)+ab(a-b)\}\{a(c-a)(a-b)+b(a-b)(b-c)$
$+c(b-c)(c-a)\} = 9abc(b-c)(c-a)(a-b)$.

Now $\Sigma bc(b-c) = -(b-c)(c-a)(a-b)$; hence we have to shew that

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)-9abc=0,$$

that is $$\Sigma a^3 - \Sigma a^2(b+c) - 6abc = 0;$$

or, since $a+b+c=0$, $2\Sigma a^3 - 6abc = 0$,

that is $$2(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=0.$$

63. Multiply the first equation by l and subtract the second; then

$$\frac{lmn-1}{1+n+mn} = l-1,$$

whence $$1+l+ln = 1+n+mn \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

Hence from the second equation

$$(1+l)(1+m+lm)+lm(1+l+ln) = (1+l+ln)(1+m+ml),$$

whence $$l(m+ml) = l(n+mn).$$

Hence $$1+m+ml = 1+n+mn = 1+l+ln \text{ or } l=0.$$

But, if $l=0$ we have from (i) $n=0$ or $m+1=0$; and if $l=n=0$ then from the first equation $m=0$; also, if $l=0=m+1$ the second denominator would be zero.

Hence the three denominators are all equal or else $l=m=n=0$; and if the three denominators are equal we have

$$1+m+mn=1+n+nm=1+m+lm=1+l+ln,$$

from which it follows that $l=m=n$.

64. $a=1-(1-a)$; $\therefore\ a+(1-a)\,b=1-(1-a)\,(1-b)$;

$$\therefore\ a+(1-a)\,b+(1-a)\,(1-b)\,c=1-(1-a)\,(1-b)\,(1-c);$$

and so on.

65. Put $b=2a$, $c=3a$, &c. in 64; then

$$1-\{(1-a)\,(1-2a)\,(1-3a)\ldots(1-na)\}=a+(1-a)\,2a$$
$$+(1-a)\,(1-2a)\,3a+\ldots+\{(1-a)\,(1-2a)\ldots(1-\overline{n-1}a)\,na\}.$$

66. This is the same as 64 with a^n for a, a^{n-1} for b, and so on.

The first two terms from the *end*

$$=\{(1-a^n)\,(1-a^{n-1})\ldots(1-a^2)\}\,\{(1-a)+a\}=\{(1-a^n)\ldots(1-a^2)\}.$$

Adding the next term from the end, we have

$$\{(1-a^n)\,(1-a^{n-1})\ldots(1-a^3)\};$$

nence proceeding in this way we get for the whole sum

$$(1-a^n)\,(1-a^{n-1})\,(1-a^{n-2})+a^{n-2}\,(1-a^n)\,(1-a^{n-1})+a^{n-1}\,(1-a^n)+a^n$$
$$=(1-a^n)\,(1-a^{n-1})+a^{n-1}\,(1-a^n)+a^n=1-a^n+a^n=1.$$

67. Let $F_n=\dfrac{1-a^n}{1-a}+\dfrac{(1-a^n)\,(1-a^{n-1})}{1-a^2}+\ldots+\dfrac{(1-a^n)\ldots(1-a)}{1-a^n}$.

Then $F_{n+1}=\dfrac{1-a^{n+1}}{1-a}+\dfrac{(1-a^{n+1})\,(1-a^n)}{1-a^2}+\ldots$

$$+\frac{(1-a^{n+1})\,(1-a^n)\ldots(1-a^2)}{1-a^n}+\frac{(1-a^{n+1})\,(1-a^n)\ldots(1-a)}{1-a^{n+1}}.$$

Hence $F_{n+1}-F_n=a^n+a^{n-1}\,(1-a^n)+a^{n-2}\,(1-a^n)\,(1-a^{n-1})+\ldots$

$+a\{(1-a^n)\,(1-a^{n-1})\ldots(1-a^2)\}+\{(1-a^n)\,(1-a^{n-1})\ldots(1-a)\}=1$, by 67.

Hence $F_{n+1}=1+F_n$.

Now $F_1=1$ and $F_2=2$.

Hence $F_3=1+F_2=3$, $F_4=1+F_3=4$, and so on indefinitely, so that $F_n=n$.

68. We have $(ax+by+cz+du)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=0$;

$$\therefore x+y+z+u+ax\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)+by\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{a}\right)+cz\left(\frac{1}{d}+\frac{1}{a}+\frac{1}{b}\right)$$
$$+du\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0;$$

$$\therefore \Sigma a^2x\,(bc+cd+db)=0;$$

$$\therefore \Sigma a^2x\,\{(b+c+d)^2-b^2-c^2-d^2\}=0;$$

$$\therefore \Sigma a^2x\,(a^2-b^2-c^2-d^2)=0;$$

$$\therefore 2\Sigma a^4x=\Sigma a^2x\,(a^2+b^2+c^2+d^2).$$

69. Since $2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$,

we have
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2n}-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)$$
$$=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2n}-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right),$$

that is
$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}.$$

70. We have $uv=fu+fv$,

$$uv=fu+fv+f(a-b)-av+bu+ab,$$

and
$$uv=fu+fv+f(a'-b')-a'v+b'u+a'b'.$$

Hence $av-bu=ab-f(b-a)$ and $a'v-b'u=a'b'-f(b'-a')$;

$$\therefore v=\frac{bb'(a-a')+f(ab'-a'b)}{ab'-a'b} \text{ and } u=\frac{aa'(b-b')+f(ab'-a'b)}{ab'-a'b}.$$

Hence $\frac{1}{f}=\frac{ab'-a'b}{aa'(b-b')+f(ab'-a'b)}+\frac{ab'-a'b}{bb'(a-a')+f(ab'-a'b)}$;

$$\therefore f^2(ab'-a'b)^2+f(ab'-a'b)\{aa'(b-b')+bb'(a-a')\}+aa'bb'(a-a')(b-b')$$
$$=(ab'-a'b)[f\{aa'(b-b')+bb'(a-a')\}+2f^2(ab'-a'b)];$$

$$\therefore f^2(ab'-a'b)^2=aa'bb'(a-a')(b-b').$$

XVII. Pages 212—213.

1. $a^{\frac{2}{3}-\frac{1}{2}}b^{\frac{5}{6}-\frac{2}{3}}=a^{\frac{1}{6}}b^{\frac{1}{6}}$.

2. $a^{\frac{2}{3}}\times a^{-\frac{3}{4}}\times a^{-\frac{2}{6}}\times a^{+\frac{5}{12}}=a^{\frac{2}{3}-\frac{3}{4}-\frac{2}{6}+\frac{5}{12}}=a^0=1$.

3. $a^{\frac{1}{2}}b^{-1}c^{\frac{3}{2}} \times a^{\frac{3}{2}}b^{\frac{2}{3}}c^{-1} = a^{\frac{1}{2}+1}b^{-1+\frac{2}{3}}c^{\frac{3}{2}-1} = a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{\frac{1}{2}}$.

4. $x^{\frac{b+c}{(c-a)(a-b)}} \times x^{\frac{c+a}{(a-b)(b-c)}} \times x^{\frac{a+b}{(b-c)(c-a)}} = x^{\frac{b+c}{(c-a)(a-b)} + \frac{c+a}{(a-b)(b-c)} + \frac{a+b}{(b-c)(c-a)}} = x^0 = 1$.

5. $(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}}) = x + x^{\frac{2}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} - x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - y = x - y$.

6. $x^1 + x^2 + 1 - (x^2 + 1 + x^{-2}) + 1 + x^{-2} + x^{-4} = x^4 + 1 + x^{-4}$.

7.
$$\begin{array}{l} x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}z^{\frac{1}{3}} + y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}} \\ \underline{x^{\frac{1}{3}} + \quad y^{\frac{1}{3}} + \quad z^{\frac{1}{3}}} \\ x \quad - x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{2}{3}}z^{\frac{1}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} - \quad x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + x^{\frac{1}{3}}z^{\frac{2}{3}} \\ \quad + x^{\frac{2}{3}}y^{\frac{1}{3}} \qquad - x^{\frac{1}{3}}y^{\frac{2}{3}} - \quad x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} \qquad + y - y^{\frac{2}{3}}z^{\frac{1}{3}} + y^{\frac{1}{3}}z^{\frac{2}{3}} \\ \qquad + x^{\frac{2}{3}}z^{\frac{1}{3}} \qquad - \quad x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} - x^{\frac{1}{3}}z^{\frac{2}{3}} \qquad + y^{\frac{2}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{2}{3}} + z \\ \hline x \qquad\qquad - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} \qquad + y \qquad\qquad + z \end{array}$$

8.
$$\begin{array}{l} x^{\frac{2}{3}} - x^{-\frac{2}{3}} \Big) x^{\frac{4}{3}} - 2 + x^{-\frac{4}{3}} \Big(x^{\frac{2}{3}} - x^{-\frac{2}{3}} \\ \underline{x^{\frac{4}{3}} - 1} \\ \quad - 1 + x^{-\frac{4}{3}} \\ \quad \underline{- 1 + x^{-\frac{4}{3}}} \end{array}$$

9.
$$\begin{array}{l} a^{\frac{1}{10}} - x^{\frac{1}{5}} \Big) a^{\frac{1}{2}} - x \quad \Big(a^{\frac{4}{10}} + a^{\frac{3}{10}}x^{\frac{1}{5}} + a^{\frac{2}{10}}x^{\frac{2}{5}} + a^{\frac{1}{10}}x^{\frac{3}{5}} + x^{\frac{4}{5}} \\ \underline{a^{\frac{1}{2}} - a^{\frac{4}{10}}x^{\frac{1}{5}}} \\ \qquad a^{\frac{4}{10}}x^{\frac{1}{5}} \\ \qquad \underline{a^{\frac{4}{10}}x^{\frac{1}{5}} - a^{\frac{3}{10}}x^{\frac{2}{5}}} \\ \qquad\qquad a^{\frac{3}{10}}x^{\frac{2}{5}} \\ \qquad\qquad \underline{a^{\frac{3}{10}}x^{\frac{2}{5}} - a^{\frac{2}{10}}x^{\frac{3}{5}}} \\ \qquad\qquad\qquad a^{\frac{2}{10}}x^{\frac{3}{5}} \\ \qquad\qquad\qquad \underline{a^{\frac{2}{10}}x^{\frac{3}{5}} - a^{\frac{1}{10}}x^{\frac{4}{5}}} \\ \qquad\qquad\qquad\qquad a^{\frac{1}{10}}x^{\frac{4}{5}} - x \\ \qquad\qquad\qquad\qquad \underline{a^{\frac{1}{10}}x^{\frac{4}{5}} - x} \end{array}$$

10.
$$\begin{array}{l} x^{\frac{1}{2}} - y^{\frac{1}{2}} \Big) x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}} \Big(x + y \\ \underline{x^{\frac{3}{2}} - xy^{\frac{1}{2}}} \\ \qquad\qquad x^{\frac{1}{2}}y - y^{\frac{3}{2}} \\ \qquad\qquad \underline{x^{\frac{1}{2}}y - y^{\frac{3}{2}}} \end{array}$$

11. $(x^{\frac{1}{6}} - 2x^{\frac{1}{2}} + x^{\frac{1}{3}})^2 = x^{\frac{2}{3}} + 4x + x^{\frac{2}{3}} + 2(-2x^{\frac{1}{2}})x^{\frac{1}{3}} + 2x^{\frac{1}{3}} \cdot x^{\frac{1}{6}} + 2x^{\frac{1}{6}}(-2x^{\frac{1}{2}})$
$= x^{\frac{2}{3}} - 4x^{\frac{4}{3}} + 2x^{\frac{7}{6}} + 4x - 4x^{\frac{5}{6}} + x^{\frac{2}{3}}$.

12.

$$
\begin{array}{l}
\qquad\qquad 4-5-4-7+6 \\
\qquad\qquad 3-4+2 \\
\hline
\qquad 12-15-12-21+18 \\
\qquad\quad -16+20+16+28-24 \\
\qquad\qquad\quad 8-10-8-14+12 \\
\hline
3-10+10-4\,)\,12-31+16-15+38-38+12\,(\,4+3+2-3 \\
\qquad 12-40+40-16 \\
\hline
\qquad\quad 9-24+1 \\
\qquad\quad 9-30+30-12 \\
\hline
\qquad\qquad 6-29+50 \\
\qquad\qquad 6-20+20-8 \\
\hline
\qquad\qquad\quad -9+30-30 \\
\qquad\qquad\quad -9+30-30+12
\end{array}
$$

Ans. $4x^2+3x+2-3x^{-1}$.

13.

$$
\begin{array}{l}
x^{\frac{1}{3}}-x^{-\frac{1}{3}}\,)\,x+2x^{\frac{5}{6}}-2x^{\frac{1}{3}}+2x^{-\frac{1}{6}}-2x^{-\frac{5}{6}}-x^{-1}\,(\,x^{\frac{2}{3}}+2x^{\frac{1}{2}}+1+2x^{-\frac{1}{2}}+x^{-\frac{2}{3}} \\
\qquad x-x^{\frac{1}{3}} \\
\hline
\qquad 2x^{\frac{5}{6}}+x^{\frac{1}{3}} \\
\qquad 2x^{\frac{5}{6}}-2x^{\frac{1}{3}} \\
\hline
\qquad\qquad x^{\frac{1}{3}} \\
\qquad\qquad x^{\frac{1}{3}}-x^{-\frac{1}{3}} \\
\hline
\qquad\qquad\quad 2x^{-\frac{1}{6}}+x^{-\frac{1}{3}} \\
\qquad\qquad\quad 2x^{-\frac{1}{6}}-2x^{-\frac{5}{6}} \\
\hline
\qquad\qquad\qquad\quad x^{-\frac{1}{3}}-x^{-1} \\
\qquad\qquad\qquad\quad x^{-\frac{1}{3}}-x^{-1}
\end{array}
$$

14.

$$
\begin{array}{l}
a^{\frac{1}{3}}x^{-\frac{1}{3}}-1+a^{-\frac{1}{3}}x^{\frac{1}{3}}\,) \\
\qquad ax^{-1} \qquad\qquad\qquad\qquad +2+a^{-1}x\,(\,a^{\frac{2}{3}}x^{-\frac{2}{3}}+a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}+a^{-\frac{2}{3}}x^{\frac{2}{3}} \\
\qquad ax^{-1}-a^{\frac{2}{3}}x^{-\frac{2}{3}}+a^{\frac{1}{3}}x^{-\frac{1}{3}} \\
\hline
\qquad\quad a^{\frac{2}{3}}x^{-\frac{2}{3}}-a^{\frac{1}{3}}x^{-\frac{1}{3}} \\
\qquad\quad a^{\frac{2}{3}}x^{-\frac{2}{3}}-a^{\frac{1}{3}}x^{-\frac{1}{3}}+1 \\
\hline
\qquad\qquad\qquad 1 \\
\qquad\qquad\qquad 1-a^{-\frac{1}{3}}x^{\frac{1}{3}}+a^{-\frac{2}{3}}x^{\frac{2}{3}} \\
\hline
\qquad\qquad\qquad\quad a^{-\frac{1}{3}}x^{\frac{1}{3}}-a^{-\frac{2}{3}}x^{\frac{2}{3}} \\
\qquad\qquad\qquad\quad a^{-\frac{1}{3}}x^{\frac{1}{3}}-a^{-\frac{2}{3}}x^{\frac{2}{3}}+a^{-1}x
\end{array}
$$

15.

$$
\frac{x^{\frac{14}{3}}+y^{\frac{28}{5}}}{x^{\frac{7}{3}}y^{\frac{14}{5}}} \div \frac{x^{\frac{7}{3}}+y^{\frac{14}{5}}}{x^{\frac{1}{3}}y^{\frac{2}{5}}}
$$

$$
=\{x^{\frac{13}{3}}-x^{\frac{10}{3}}y^{\frac{4}{5}}+x^{\frac{8}{3}}y^{\frac{8}{5}}-x^{\frac{6}{3}}y^{\frac{12}{5}}+x^{\frac{4}{3}}y^{\frac{16}{5}}-x^{\frac{2}{3}}y^{\frac{20}{5}}+y^{\frac{24}{5}}\}\,x^{-\frac{6}{3}}y^{-\frac{12}{5}}
$$

$$
=x^2y^{-\frac{12}{5}}-x^{\frac{4}{3}}y^{-\frac{8}{5}}+x^{\frac{2}{3}}y^{-\frac{4}{5}}-1+\ldots\ldots
$$

16. Left side $= \dfrac{x-1}{x^{\frac{1}{3}}-1} - \dfrac{x^{\frac{2}{3}}-1}{x^{\frac{1}{3}}+1} = x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1 - (x^{\frac{1}{3}}-1) = x^{\frac{2}{3}} + 2.$

17. $(2x+y^{-1})(2y+x^{-1}) = 4xy + 4 + x^{-1}y^{-1} = (2x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y^{-\frac{1}{2}})^2.$

18. $\dfrac{a^2+b^2-a^{-2}-b^{-2}}{a^2b^2-a^{-2}b^{-2}} = \dfrac{a^2b^2(a^2+b^2-a^{-2}-b^{-2})}{a^2b^2(a^2b^2-a^{-2}b^{-2})} = \dfrac{(a^2b^2-1)(a^2+b^2)}{a^4b^4-1}$

$= \dfrac{a^2+b^2}{a^2b^2+1}$; also $\dfrac{(a-a^{-1})(b-b^{-1})}{ab+a^{-1}b^{-1}} = \dfrac{(a^2-1)(b^2-1)}{a^2b^2+1}$.

Hence the sum $= \dfrac{a^2+b^2+(a^2-1)(b^2-1)}{a^2b^2+1} = 1.$

19. We have $x^{\frac{1}{3}}+y^{\frac{1}{3}} = -z^{\frac{1}{3}}$, or by cubing

$x+y+3x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{1}{3}}+y^{\frac{1}{3}}) = -z$; $\therefore\ x+y+z = -3x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{1}{3}}+y^{\frac{1}{3}}) = 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$.

Hence $(x+y+z)^3 = (3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}})^3 = 27xyz.$

20. (i) Put $a^{\frac{1}{2}} = X$ and $b^{\frac{1}{3}} = Y$; then X^6 and Y^6 are both rational; and we have

$$(X+Y)(X^5 - X^4Y + X^3Y^2 - X^2Y^3 + XY^4 - Y^5) = X^6 - Y^6.$$

Hence the factor required is $X^5 - X^4Y + \ldots\ldots$, i.e.

$$a^{\frac{5}{2}} - a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - \ldots\ldots - b^{\frac{20}{3}}.$$

(ii) Put $a^{\frac{3}{2}}x^{\frac{5}{6}} = X$ and $y^{\frac{1}{2}} = Y$; then X^6 and Y^6 are both rational. Hence, as in (i) the required rational factor is $X^5 - X^4Y + \ldots\ldots - Y^5$, that is

$$a^{\frac{15}{2}}x^{\frac{25}{6}} - a^{\frac{9}{2}}x^{\frac{20}{6}}y^{\frac{1}{2}} + \ldots\ldots - y^{\frac{5}{2}}.$$

(iii) We know that

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab) = a^3+b^3+c^3-3abc.$$

Hence $(a+bx^{\frac{1}{3}}+cx^{\frac{2}{3}})(a^2+b^2x^{\frac{2}{3}}+c^2x^{\frac{4}{3}}-bcx-cax^{\frac{2}{3}}-abx^{\frac{1}{3}})$

$= a^3 + (bx^{\frac{1}{3}})^3 + (cx^{\frac{2}{3}})^3 - 3a \cdot bx^{\frac{1}{3}} \cdot cx^{\frac{2}{3}} = a^3 + b^3x + c^3x^2 - 3abcx.$

Hence the required factor is

$$a^2 + b^2x^{\frac{2}{3}} + c^2x^{\frac{4}{3}} - bcx - cax^{\frac{2}{3}} - abx^{\frac{1}{3}}.$$

(iv) As in (iii), $\Sigma x^{\frac{1}{3}}(\Sigma x^{\frac{2}{3}} - \Sigma y^{\frac{1}{3}}z^{\frac{1}{3}}) = (x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}})$.

Also $(A-B^{\frac{1}{3}})(A^2+AB^{\frac{1}{3}}+B^{\frac{2}{3}}) = A^3 - B.$

Hence $\Sigma x^{\frac{1}{3}}(\Sigma x^{\frac{2}{3}} - \Sigma y^{\frac{1}{3}}z^{\frac{1}{3}})[(x+y+z)^2 + 3(x+y+z)(xyz)^{\frac{1}{3}} + 9(xyz)^{\frac{2}{3}}]$

$= (x+y+z)^3 - 27xyz.$

The factor required is therefore

$$\{\Sigma x^{\frac{2}{3}} - \Sigma y^{\frac{1}{3}}z^{\frac{1}{3}}\}\{(x+y+z)^2 + 3(x+y+z)x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + 9x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}\}.$$

21. As in 19, we have

$$\{(1-x^3)(y-z)^3+(1-y^3)(z-x)^3+(1-z^3)(x-y)^3\}^3$$
$$=27(y-z)^3(z-x)^3(x-y)^3(1-x^3)(1-y^3)(1-z^3).$$

Now $\Sigma(1-x^3)(y-z)^3=\Sigma(y-z)^3-\Sigma x^3(y-z)^3$

$$=3(y-z)(z-x)(x-y)+3\Sigma x^3yz(y-z)$$
$$=3(y-z)(z-x)(x-y)(1-xyz).$$

Hence $27(y-z)^3(z-x)^3(x-y)^3(1-xyz)^3$

$$=27(y-z)^3(z-x)^3(x-y)^3(1-x^3)(1-y^3)(1-z^3).$$

XVIII. Pages 220–221.

1. $\dfrac{\sqrt3-1}{\sqrt3+1}=\dfrac{(\sqrt3-1)(\sqrt3-1)}{(\sqrt3+1)(\sqrt3-1)}=\dfrac{4-2\sqrt3}{3-1}=2-\sqrt3.$

2. $\dfrac{2\sqrt5}{\sqrt5+\sqrt3}=\dfrac{2\sqrt5(\sqrt5-\sqrt3)}{(\sqrt5+\sqrt3)(\sqrt5-\sqrt3)}=\dfrac12(10-2\sqrt{15}).$

3. $\dfrac{1}{\sqrt8+\sqrt3}+\dfrac{1}{\sqrt8-\sqrt3}=\dfrac{\sqrt8-\sqrt3}{5}+\dfrac{\sqrt8+\sqrt3}{5}=\dfrac25\sqrt8=\dfrac45\sqrt2.$

4. $\dfrac{(2+\sqrt3)^3}{(2-\sqrt3)^3(2+\sqrt3)^3}+\dfrac{(2-\sqrt3)^3}{(2+\sqrt3)^3(2-\sqrt3)^3}=(2+\sqrt3)^3+(2-\sqrt3)^3=52.$

5. $\dfrac{3\sqrt2(\sqrt6-\sqrt3)}{(\sqrt6+\sqrt3)(\sqrt6-\sqrt3)}-\dfrac{4\sqrt3(\sqrt6-\sqrt2)}{(\sqrt6+\sqrt2)(\sqrt6-\sqrt2)}+\dfrac{\sqrt6(\sqrt3-\sqrt2)}{(\sqrt3+\sqrt2)(\sqrt3-\sqrt2)}$
$=\sqrt{12}-\sqrt6-\sqrt{18}+\sqrt6+\sqrt{18}-\sqrt{12}=0.$

6. $(7-2\sqrt5)(31+13\sqrt5)=217-130+29\sqrt5=29(3+\sqrt5);$

also $(6-2\sqrt7)(11+4\sqrt7)=66-56+2\sqrt7=2(5+\sqrt7).$

Hence the fraction $=\dfrac{29}{2}.$

7. $\dfrac{\sqrt2+\sqrt3-\sqrt5}{(\sqrt2+\sqrt3)^2-5}=\dfrac{1}{2\sqrt6}(\sqrt2+\sqrt3-\sqrt5)=\dfrac{1}{12}(2\sqrt3+3\sqrt2-\sqrt{30}).$

8. $\dfrac{\sqrt3-\sqrt5-\sqrt2}{(\sqrt3-\sqrt2)^2-5}=\dfrac{1}{2\sqrt6}(\sqrt5+\sqrt2-\sqrt3)=\dfrac{1}{12}(\sqrt{30}+2\sqrt3-3\sqrt2).$

9. $\dfrac{1}{\sqrt{10}+\sqrt{14}+\sqrt{15}+\sqrt{21}}=\dfrac{1}{(\sqrt2+\sqrt3)(\sqrt5+\sqrt7)}$

$$=\frac{(\sqrt3-\sqrt2)(\sqrt7-\sqrt5)}{(3-2)(7-5)}=\frac12(\sqrt{21}+\sqrt{10}-\sqrt{14}-\sqrt{15}).$$

10. $\dfrac{1}{\sqrt{6}+\sqrt{21}-\sqrt{10}-\sqrt{35}} = \dfrac{1}{(\sqrt{3}-\sqrt{5})(\sqrt{2}+\sqrt{7})}$

$= \frac{1}{10}(\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{7}) = \frac{1}{10}(\sqrt{6}+\sqrt{10}-\sqrt{21}-\sqrt{35})$.

11. $\dfrac{\sqrt[3]{4}+\sqrt[3]{2}+1}{2-1} + \dfrac{\sqrt[3]{4}-\sqrt[3]{2}+1}{2+1} = \frac{1}{3}\{4\sqrt[3]{4}+2\sqrt[3]{2}+4\}$.

12. $\dfrac{4(\sqrt[3]{81}+\sqrt[3]{9}+1)}{9-1} + \dfrac{5(\sqrt[3]{81}-\sqrt[3]{9}+1)}{9+1} = \sqrt[3]{81}+1$.

13. $\dfrac{1-\sqrt[3]{2}}{(1-\sqrt[3]{2})(1+\sqrt[3]{2}+\sqrt[3]{4})} = \dfrac{1-\sqrt[3]{2}}{1-2} = \sqrt[3]{2}-1$.

14. $\dfrac{1}{\sqrt[3]{2}(1+\sqrt[3]{3}+\sqrt[3]{9})} = \dfrac{\sqrt[3]{4}(1-\sqrt[3]{3})}{2(1-3)} = \frac{1}{4}(\sqrt[3]{12}-\sqrt[3]{4})$.

15. Put $\sqrt{(101-28\sqrt{13})} = \sqrt{x}-\sqrt{y}$; then $101-28\sqrt{13} = x+y-2\sqrt{xy}$. Hence $x+y=101$ and $xy=14^2 \cdot 13$, whence $x=52$ and $y=49$. Thus

$$\sqrt{(101-28\sqrt{13})} = \sqrt{52}-7.$$

16. Put $\sqrt{(28-5\sqrt{12})} = \sqrt{x}-\sqrt{y}$; then $28=x+y$ and $xy=75$, whence $x=25$, $y=3$, and therefore $\sqrt{(28-5\sqrt{12})} = 5-\sqrt{3}$.

17. $\sqrt{\{11+2(1+\sqrt{5})(1+\sqrt{7})\}} = \{13+2\sqrt{5}+2\sqrt{7}+2\sqrt{35}\}$
$= \sqrt{x}+\sqrt{y}+\sqrt{z}$.

Then we are to find, *if possible*, values of x, y, z which satisfy simultaneously

$$x+y+z=13,\ yz=35,\ zx=7,\ xy=5.$$

The only values which satisfy the last three are

$$\frac{x}{1} = \frac{y}{5} = \frac{z}{7} = \pm 1, \text{ and } 1+5+7=13.$$

Hence the required root is $1+\sqrt{5}+\sqrt{7}$.

18. $\sqrt{\{6-4\sqrt{3}+\sqrt{(16-8\sqrt{3})}\}}$. First $\sqrt{(16-8\sqrt{3})} = 2\sqrt{3}-2$. Then $\sqrt{\{6-4\sqrt{3}+2\sqrt{3}-2\}} = \sqrt{\{4-2\sqrt{3}\}} = \sqrt{3}-1$.

19. $\sqrt[3]{(97-56\sqrt{3})} = \sqrt{x}-\sqrt{y}$, where $x+y=97$ and $xy=3\times 28^2$. Hence $x=49$, $y=48$. Hence $\sqrt[3]{(97-56\sqrt{3})} = \sqrt{(7-4\sqrt{3})}$, which will be found to be $2-\sqrt{3}$.

20. $\sqrt{(3+2\sqrt{2})} = \sqrt{2}+1$ and $\sqrt{(3-2\sqrt{2})} = \sqrt{2}-1$; hence the fraction

$$= \frac{1}{2\sqrt{2}-1} = \frac{1}{7}(2\sqrt{2}+1).$$

21. $\sqrt{(7-2\sqrt{10})} = \sqrt{5}-\sqrt{2}$. Hence we have

$$\frac{\sqrt{2}+\sqrt{45}}{\sqrt{2}+\sqrt{5}-\sqrt{2}} = \frac{1}{\sqrt{5}}(\sqrt{2}+3\sqrt{5}) = 3+\frac{1}{5}\sqrt{10}.$$

22. $\sqrt{(2+\sqrt{3})} = \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}} = \frac{\sqrt{3}+1}{\sqrt{2}}$. Hence we have

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\frac{\sqrt{3}+1}{\sqrt{2}}} - \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-\frac{\sqrt{3}+1}{\sqrt{2}}} = \frac{\sqrt{6}+2}{3+\sqrt{3}} - \frac{\sqrt{6}-2}{1-\sqrt{3}}$$

$$= \frac{1}{6}(\sqrt{6}+2)(3-\sqrt{3}) + \frac{1}{2}(\sqrt{6}-2)(\sqrt{3}+1) = \sqrt{2}+\sqrt{6}-\frac{4}{3}\sqrt{3}.$$

23. $\sqrt{(5\pm 2\sqrt{6})} = \sqrt{3}\pm\sqrt{2}$. Hence we have

$$\frac{\sqrt{3}+\sqrt{2}-(\sqrt{3}-\sqrt{2})}{\sqrt{3}+\sqrt{2}+(\sqrt{3}-\sqrt{2})} = \frac{\sqrt{2}}{\sqrt{3}}.$$

24. Let $\sqrt{(6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6})} = \sqrt{x}+\sqrt{y}+\sqrt{z}$; then we must have $x+y+z=6$ and $yz=6$, $zx=3$, $xy=2$ simultaneously true; and these equations are all satisfied by the values $x=1$, $y=2$, $z=3$. Hence the required root is

$$1+\sqrt{2}+\sqrt{3}.$$

25. Let $\sqrt{(11+6\sqrt{2}+4\sqrt{3}+2\sqrt{6})} = \sqrt{(11+2\sqrt{18}+2\sqrt{12}+2\sqrt{6})}$
$= \sqrt{x}+\sqrt{y}+\sqrt{z}$;

then we must have the equations $x+y+z=11$, $yz=18$, $zx=12$, $xy=6$ simultaneously satisfied; and these equations are satisfied by the values $x=2$, $y=3$ and $z=6$. Hence the required root is $\sqrt{2}+\sqrt{3}+\sqrt{6}$.

26. Assume

$$\sqrt{(17+4\sqrt{2}-4\sqrt{3}-4\sqrt{6}-4\sqrt{5}-2\sqrt{10}+2\sqrt{30})} = \sqrt{x}+\sqrt{y}-\sqrt{z}-\sqrt{w}.$$

Then $17+2\sqrt{8}+2\sqrt{30}-2\sqrt{12}-2\sqrt{24}-2\sqrt{20}-2\sqrt{10}$

$$= x+y+z+w+2\sqrt{xy}+2\sqrt{zw}-2\sqrt{xz}-2\sqrt{xw}-2\sqrt{yz}-2\sqrt{yw}.$$

Now if we take $xy=30$ and $zw=8$, the values $x=6$, $y=5$, $z=2$, $w=4$ satisfy these equations and also the equation $x+y+z+w=17$. It will now be found that

$$(\sqrt{6}+\sqrt{5}-\sqrt{2}-2)^2 = 17+4\sqrt{2}-4\sqrt{3}-4\sqrt{6}-4\sqrt{5}-2\sqrt{10}+2\sqrt{30}.$$

27. $\sqrt{(12-\sqrt{140})} = \sqrt{7}-\sqrt{5}$, $\sqrt{(8-\sqrt{60})} = \sqrt{5}-\sqrt{3}$

and $\sqrt{(10+2\sqrt{21})} = \sqrt{7}+\sqrt{3}$. Hence we have

$$\frac{1}{\sqrt{7}-\sqrt{5}} - \frac{1}{\sqrt{5}-\sqrt{3}} - \frac{2}{\sqrt{7}+\sqrt{3}}$$

$$= \frac{1}{2}(\sqrt{7}+\sqrt{5}) - \frac{1}{2}(\sqrt{5}+\sqrt{3}) - \frac{1}{2}(\sqrt{7}-\sqrt{3}) = 0.$$

28. We have

$$\frac{1}{\sqrt{6}-\sqrt{5}} - \frac{3}{\sqrt{5}-\sqrt{2}} - \frac{4}{\sqrt{6}+\sqrt{2}} = \sqrt{6}+\sqrt{5}-(\sqrt{5}+\sqrt{2})-(\sqrt{6}-\sqrt{2}) = 0,$$

XIX. Pages 238–240.

1. $\sqrt{4x^{10}} = 2x^5$ and $\sqrt{9y^6} = 3y^3$. Hence the required root is $2x^5 - 3y^3$.

2. $\sqrt{x^8} = x^4$ and $\sqrt{9x^4y^{12}} = 3x^2y^6$. Hence the required root is $x^4 - 3x^2y^6$.

3. $a^2 - 2a(2b+3c) + 4b^2 + 9c^2 + 12bc = a^2 - 2a(2b+3c) + (2b+3c)^2$. Hence the required root is $a - 2b - 3c$.

4. We have $(5a^2)^2 - 2(5a^2)(3b^2+2c^2) + (3b^2+2c^2)^2 = (5a^2 - 3b^2 - 2c^2)^2$.

5.
$$x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1 \,\big(\, x^3 + x^2 + x + 1$$
$$(x^3)^2 = x^6$$
$$(x^3 + x^2)^2 = x^6 + 2x^5 + x^4$$
$$(x^3 + x^2 + x)^2 = x^6 + 2x^5 + 3x^4 + 2x^3 + x^2$$
$$(x^3 + x^2 + x + 1)^2 = x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$$

6.
$$4x^4 - 8x^3y^2 + 4xy^6 + y^8 \,\big(\, 2x^2 - 2xy^2 - y^4$$
$$(2x^2)^2 = 4x^4$$
$$(2x^2 - 2xy^2)^2 = 4x^4 - 8x^3y^2 + 4x^2y^4$$
$$(2x^2 - 2xy^2 - y^4)^2 = 4x^4 - 8x^3y^2 + 4xy^6 + y^8$$

7.
$$49 + 112x^2 + 70x^3 + 64x^4 + 80x^5 + 25x^6 \,\big(\, 7 + 8x^2 + 5x^3$$
$$7^2 = 49$$
$$(7 + 8x^2)^2 = 49 + 112x^2 + 64x^4$$
$$(7 + 8x^2 + 5x^3)^2 = 49 + 112x^2 + 70x^3 + 64x^4 + 80x^5 + 25x^6$$

8.
$$x^4 - 2x^3 + 5x^2 - 6x + 8 - 6x^{-1} + 5x^{-2} - 2x^{-3} + x^{-4} \,\big(\, x^2 - x + 2 - x^{-1} + x^{-2}$$
$$(x^2)^2 = x^4$$
$$(x^2 - x)^2 = x^4 - 2x^3 + x^2$$
$$(x^2 - x + 2)^2 = x^4 - 2x^3 + 5x^2 - 4x + 4$$
$$(x^2 - x + 2 - x^{-1})^2 = x^4 - 2x^3 + 5x^2 - 6x + 6 - 4x^{-1} + x^{-2}$$
$$(x^2 - x + 2 - x^{-1} + x^{-2})^2 = x^4 - 2x^3 + 5x^2 - 6x + 8 - 6x^{-1} + 5x^{-2} - 2x^{-3} + x^{-4}$$

9. We have $\{25^2x^4 - 500x^3y + 50x^2y^2 + 20xy^3 + y^4\} / 25x^2y^2$.
$$25^2x^4 - 500x^3y + 50x^2y^2 + 20xy^3 + y^4 \,\big(\, 25x^2 - 10xy - y^2$$
$$(25x^2)^2 = 25^2x^4$$
$$(25x^2 - 10xy)^2 = 25^2x^4 - 500x^3y + 100x^2y^2$$
$$(25x^2 - 10xy - y^2)^2 = 25^2x^4 - 500x^3y + 50x^2y^2 + 20xy^3 + y^4$$

Hence root required $= (25x^2 - 10xy - y^2) / 5xy$.

10.

$$x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x + 4x^{\frac{2}{3}} + x^{\frac{1}{3}} \left(x^{\frac{5}{6}} - 2x^{\frac{1}{2}} - x^{\frac{1}{6}} \right.$$

$$(x^{\frac{5}{6}})^2 = x^{\frac{5}{3}}$$

$$(x^{\frac{5}{6}} - 2x^{\frac{1}{2}})^2 = x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 4x$$

$$(x^{\frac{5}{6}} - 2x^{\frac{1}{2}} - x^{\frac{1}{6}})^2 = x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x + 4x^{\frac{2}{3}} + x^{\frac{1}{3}}$$

11.

$$x^{\frac{2}{3}} - 4x^{\frac{5}{6}} + 4x + 2x^{\frac{7}{6}} - 4x^{\frac{4}{3}} + x^{\frac{5}{3}} \left(x^{\frac{1}{3}} - 2x^{\frac{1}{2}} + x^{\frac{5}{6}} \right.$$

$$(x^{\frac{1}{3}})^2 = x^{\frac{2}{3}}$$

$$(x^{\frac{1}{3}} - 2x^{\frac{1}{2}})^2 = x^{\frac{2}{3}} - 4x^{\frac{5}{6}} + 4x$$

$$(x^{\frac{1}{3}} - 2x^{\frac{1}{2}} + x^{\frac{5}{6}})^2 = x^{\frac{2}{3}} - 4x^{\frac{5}{6}} + 4x + 2x^{\frac{7}{6}} - 4x^{\frac{4}{3}} + x^{\frac{5}{3}}$$

12.

$$a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + x^{\frac{8}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{\frac{8}{5}} \left(a^{-\frac{3}{5}}x^{\frac{7}{5}} \right.$$

$$(a^{-\frac{3}{5}}x^{\frac{7}{5}})^2 = a^{-\frac{6}{5}}x^{\frac{14}{5}} \qquad -x^{\frac{4}{5}} - a^{\frac{4}{5}}$$

$$(a^{-\frac{3}{5}}x^{\frac{7}{5}} - x^{\frac{4}{5}})^2 = a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + x^{\frac{8}{5}}$$

$$(a^{-\frac{3}{5}}x^{\frac{7}{5}} - x^{\frac{4}{5}} - a^{\frac{4}{5}})^2 = a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + x^{\frac{8}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{\frac{8}{5}}$$

13, 14, 15. The cube roots of these expressions can all be written down by inspection [see Note, Art. 200]. The roots are respectively

$$x - 8, \quad x^2 - xy + y^2, \quad 1 - 3x^2 + 2x^4.$$

16. Arrange in powers of a; then we have

$$a^2 \{2(b+c)^2 + 2b^2 + 2c^2 + 4bc\} + a\{4b^2c + 4c^2b + 4bc(b+c)\} + 4b^2c^2,$$

that is

$$4(b+c)^2 a^2 + 8abc(b+c) + 4b^2c^2,$$

which is obviously

$$\{2(b+c)a + 2bc\}^2.$$

17. Arrange in powers of x; then we have

$$x^4 - 2x^3(y+z) + x^2(y^2+z^2) + 2xyz(y+z) + y^2z^2.$$

$$x^4 - 2x^3(y+z) + x^2(y^2+z^2) + 2xyz(y+z) + y^2z^2 \left(x^2 - x(y+z) - yz \right.$$

$$(x^2)^2 = x^4$$

$$\{x^2 - x(y+z)\}^2 = x^4 - 2x^3(y+z) + x^2(y+z)^2$$

$$\{x^2 - x(y+z) - yz\}^2 = x^4 - 2x^3(y+z) + x^2(y+z)^2 + 2xyz(y+z) + y^2z^2$$

18. Arrange in powers of a; then we have

$$a^4 + 2a^2b^2 + b^4 = (a^2+b^2)^2.$$

19. $(x+a)(x+2a)(x+3a)(x+4a) + a^4 = (x^2+5ax+4a^2)(x^2+5ax+6a^2) + a^4$

$$= (x^2+5ax)^2 + 10a^2(x^2+5ax) + 25a^4 = (x^2+5ax+5a^2)^2.$$

20. $$x^4+px^3+qx^2+rx+s\left(x^2+\frac{p}{2}x+\frac{1}{2}\left(q-\frac{p^2}{4}\right)\right.$$

$$(x^2)^2=x^4$$

$$\left(x^2+\frac{p}{2}x\right)^2=x^4+px^3+\frac{p^2}{4}x^2$$

$$\left\{x^2+\frac{p}{2}x+\frac{1}{2}\left(q-\frac{p^2}{4}\right)\right\}^2=x^4+px^3+qx^2+\frac{1}{2}p\left(q-\frac{p^2}{4}\right)x+\frac{1}{4}\left(q-\frac{p^2}{4}\right)^2.$$

Thus $\left\{x^2+\frac{p}{2}x+\frac{1}{2}\left(q-\frac{p^2}{4}\right)\right\}^2$ agrees with the given expression so far as the terms which contain x^2 and higher powers are concerned, and therefore $x^2+\frac{p}{2}x+\frac{1}{2}\left(q-\frac{p^2}{4}\right)$ is the square root of the given expression provided it is a perfect square, the conditions for which are

$$r=\frac{1}{2}p\left(q-\frac{p^2}{4}\right) \text{ and } s=\frac{1}{4}\left(q-\frac{p^2}{4}\right)^2,$$

that is $p^3-4pq+8r=0$ and $p^2s=r^2$.

21. If the given expression be a perfect square it is obvious that the first two terms are $2x^3-6x^2$ and that the last two terms are $\pm(5-4x)$. Hence the expression, if a square at all, must be

$$(2x^3-6x^2\mp4x\pm5)^2=4x^6-24x^5+(36\mp16)x^4+x^3(\pm20\pm48)+x^2(\mp60+16)-40x+25.$$

Hence $A=20$, $B=68$, $C=-44$ or $A=52$, $B=-68$, $C=76$.

22. If the expression is a perfect cube it must

$$=(x\sqrt[3]{a}+\sqrt[3]{d})^3=ax^3+3x^2\sqrt[3]{a^2d}+3x\sqrt[3]{ad^2}+d.$$

Hence $b=3\sqrt[3]{a^2d}$ and $c=3\sqrt[3]{ad^2}$, which are equivalent to $b^2=3ac$ and $c^2=3bd$.

23. If the given expression be the square of a rational and integral expression in x, y, and z, it must be

$$(x\sqrt{a}+y\sqrt{b}+z\sqrt{c})^2=ax^2+by^2+cz^2+2yz\sqrt{bc}+2zx\sqrt{ca}+2xy\sqrt{ab}.$$

Hence $f=\sqrt{bc}$, $g=\sqrt{ca}$, and $h=\sqrt{ab}$; therefore $gh=af$, $hf=bg$, and $fg=ch$.

24. As in 23, the conditions are $gh-(a-\lambda)f=0$, &c.

Hence $$a-\frac{gh}{f}=\lambda=\&c.$$

25. Suppose the expression whose cube root is to be found is the cube of

$$(a_1x^n+\ldots a_rx^{n-r+1})+(a_{r+1}x^{n-r}+\ldots a_{2r}x^{n-2r+1})+R.$$

Then the expression is equal to

$$(a_1x^n+\ldots a_rx^{n-r+1})^3+3(a_1x^n+\ldots+a_rx^{n-r+1})^2(a_{r+1}x^{n-r}+\ldots+a_{2r}x^{n-2r+1})$$
$$+[3(a_1x^n+\ldots+a_rx^{n-r+1})(a_{r+1}x^{n-r}+\ldots+a_{2r}x^{n-2r+1})^2$$
$$+(a_{r+1}x^{n-r}+\ldots+a_{2r}x^{n-2r+1})^3+3R(a_1x^n+\ldots)^2+3R^2(a_1x^n+\ldots)+R^3].$$

The coefficients a_1, a_2, a_3..., a_{2r} will be determined by equating the coefficients of the first $2r$ powers in the expression last written and in the given expression. But, since the highest power of x contained in R is x^{n-2r}, it is easy to see that the highest power of x contained in the expression within square brackets is x^{3n-2r}. Hence the terms within square brackets will not affect any of the terms from which a_1, a_2,...a_{2r} are determined, for the first $2r$ terms of the given expression extend from x^{2n} to $x^{2n-2r+1}$. It therefore follows that if the cube of the sum of the first r terms of the root be subtracted from the given expression, and the remainder be divided by three times the square of the sum of the first r terms, the quotient will give the next r terms of the root.

26. Let N be the given number, which is a perfect square of a number containing $2n+2$ figures, and let p be the number formed by the first $n+2$ figures of the root followed by n zeros, and let q be the number formed by the remaining n figures.

Then $$\sqrt{N}=p+q$$

$$\therefore \quad (N-p^3)/3p^2=q+(3pq^2+q^3)/3p^2.$$

Now $$10^{2n+2}>p\nless 10^{2n+1},$$

and $$10^n>q\nless 10^{n-1}.$$

Hence $$\frac{q^2}{p}+\frac{q^3}{3p^2}<\frac{10^{2n}}{10^{2n+1}}+\frac{10^{3n}}{3\times 10^{4n+2}}.$$

Thus $(N-p^3)/3p^2$ differs from q by a proper fraction; whence it follows that if p^3 be subtracted from N and the remainder be divided by $3p^2$, the integral part of the quotient will be q.

27. Let $x=a+b$, then since $x^3+qx-r=0$,

we have $$(a+b)^3+q(a+b)-r=0;$$

$$\therefore \quad b=(r-qa-a^3)/(3a^2+q)-(3ab^2+b^3)/(3a^2+q).$$

Now let a be the number formed by the first $n+2$ figures of the value of x, and let b be the number formed by the rest of the root. Then $a/b\nless 10^{n+1}$.

Hence $$(3ab^2+b^3)/(3a^2+q)\nless 4ab^2/3a^2$$

$$\nless \frac{4}{3\times 10^{n+1}}b.$$

Thus $(3ab^2+b^3)/(3a^2+q)$ can at most only make a difference of one in the nth place of b.

Hence $(r-qa-a^3)/(3a^2+q)$ can differ from b only in the $(n-1)$th place.

XX. Pages 253–254.

1. Since $a+b$, $b+c$, $c+a$ are in continued proportion

$$(a+b)(c+a)-(b+c)^2=0;$$

$$\therefore \quad c^2-a^2=ac+ab-bc-b^2=(a-b)(b+c).$$

Hence $$b+c : c+a=c-a : a-b.$$

2. Let $\frac{x}{a}=\lambda$, then $\frac{y}{b}=\frac{z}{c}=\lambda$.

Hence $$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\lambda^3(a+b+c);$$

and $$\frac{(x+y+z)^3}{(a+b+c)^2}=\frac{(a\lambda+b\lambda+c\lambda)^3}{(a+b+c)^2}=\lambda^3(a+b+c).$$

Hence $$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

3. We have $$(a+d)^2-(b+c)^2=(a-d)^2-(b-c)^2;$$
$$\therefore\ ad=bc;\quad \therefore\ a:b::c:d.$$

4. In order that
$$a+b+c:c+a-b=a+b-c:b+c-a$$
it is necessary and sufficient that $(b+c)^2-a^2=a^2-(b-c)^2$, or $b^2+c^2=a^2$.

5. Let x be the number; then $7-x:10-x::19-x:31-x$; hence $(7-x)(31-x)=(10-x)(19-x)$, whence $x=3$.

6. We have from Art. 211, Ex.
$$\frac{b}{a+b}=\frac{a+c}{2b+c}=\frac{a+c}{a+2c}.$$

Hence $a+c=0$, or else $a+2c=2b+c$ that is $a+c=2b$.

If $a+c=0$ we have $b=0$.

If $a+c=2b$ we have $\frac{b}{a+b}=\frac{2b}{4b-a}$, whence $b=0$ or $2b=3a$; and $2b=3a$ with $a+c=2b$ gives $c=2a$.

Thus we have $\frac{a}{2}=\frac{b}{3}=\frac{c}{4}$, excluding the relations $\frac{a}{1}=\frac{b}{0}=\frac{c}{-1}$ which require that a and c should be infinite.

7. By Art. 113 each of the equal fractions is equal to each of

$$\frac{x+y+z}{a+b+c},\quad \frac{ax+by+cz}{2\Sigma bc-\Sigma a^2}\quad\text{and}\quad \frac{\sqrt{(\Sigma yz)}}{\sqrt{\{\Sigma(c+a-b)(a+b-c)\}}}=\frac{\sqrt{(\Sigma yz)}}{\sqrt{\{2\Sigma bc-\Sigma a^2\}}}.$$

Hence $$\frac{(x+y+z)(ax+by+cz)}{(a+b+c)}=\Sigma yz.$$

8. We have $$\frac{y+z}{\frac{1}{a}}=\frac{z+x}{\frac{1}{b}}=\frac{x+y}{\frac{1}{c}}.$$

Hence each is equal to $\dfrac{y-z}{\frac{1}{c}-\frac{1}{b}} = \dfrac{z-x}{\frac{1}{a}-\frac{1}{c}} = \dfrac{x-y}{\frac{1}{b}-\frac{1}{a}}$;

$$\therefore \frac{y-z}{a(b-c)} = \frac{z-x}{b(c-a)} = \frac{x-y}{c(a-b)}.$$

9. Let $\dfrac{a_r}{b_r}$ be the greatest of the fractions and let $\dfrac{a_r}{b_r} = g$; then we have $\dfrac{a_1}{b_1} < g$ and therefore $l_1a_1 < g \,.\, l_1b_1$; so also $l_2a_2 < g \,.\, l_2b_2, \ldots, l_ra_r = g \,.\, l_rb_r, \ldots$.

Hence $\Sigma l_1a_1 < g \,.\, \Sigma l_1b_1$ and therefore $\dfrac{\Sigma l_1a_1}{\Sigma l_1b_1} < g$. This proves that $\Sigma l_1a_1 : \Sigma l_1b_1$ is less than the greatest of $a_1 : b_1$, &c., and it can be proved in a similar manner to be greater than the least.

10. We have $ad = bc$, and therefore $(abcd)^n = a^{2n}d^{2n} = b^{2n}c^{2n}$. Hence

$$\begin{aligned}(abcd)^n (a^{-2n} + b^{-2n} + c^{-2n} + d^{-2n}) \\ = a^{2n}d^{2n} \,.\, a^{-2n} + b^{2n}c^{2n} \,.\, b^{-2n} + b^{2n}c^{2n} \,.\, c^{-2n} + a^{2n}d^{2n} \,.\, d^{-2n} \\ = d^{2n} + c^{2n} + b^{2n} + a^{2n}.\end{aligned}$$

11. Arrange in powers of a; then

$$\begin{aligned}a^2(b+c)(c+d) + a(b+d)(b+c)(d+c) + bd(b+c)(c+d) \\ = a^2\{cd + db + bc\} + a\{bcd + (b+c+d)(cd+db+bc)\} + bcd(b+c+d); \\ \therefore a^2c^2 - 2abcd + b^2d^2 = 0, \text{ that is } (ac-bd)^2 = 0.\end{aligned}$$

12. The given relation may be written

$$a^3bcd - a^2(b^2c^2 + c^2d^2 + d^2b^2) + abcd(b^2 + c^2 + d^2) - b^2c^2d^2 = 0,$$

that is $$(ab-cd)(ac-bd)(ad-bc) = 0.$$

Hence a, b, c, d are proportionals when taken in some order or other.

13. Each of the given equal fractions is equal to either of

$$\frac{x+2y+z}{(a+2b+c)+2(a-c)+(a-2b+c)} = \frac{x-z}{(a+2b+c)-(a-2b+c)}$$

$$= \frac{x-2y+z}{(a+2b+c)-2(a-c)+(a-2b+c)};$$

$$\therefore \quad (x+2y+z)/a = (x-z)/b = (x-2y+z)/c.$$

14. Eliminating z we have

$$ax^2 + by^2 + c(x+y)^2 - 2fy(x+y) - 2gx(x+y) + 2hxy = 0,$$

or $$(a+c-2g)x^2 + 2(c-f-g+h)xy + (b+c-2f)y^2 = 0.$$

The condition that the values of x/y given by the above equation are equal is therefore [Art. 121]

$$(a+c-2g)(b+c-2f) - (c-f-g+h)^2 = 0,$$

which is equivalent to the result given.

15. We have

$$\frac{a}{p(px-qy-rz)}=\frac{b}{q(qy-rz-px)}=\frac{c}{r(rz-px-qy)}=\lambda;$$

$$\therefore\quad a=\lambda p(px-qy-rz),$$
$$b=\lambda q(qy-rz-px),$$
$$c=\lambda r(rz-px-qy).$$

Hence $ax-by-cz=\lambda\{(px-qy-rz)px-(qy-rz-px)qy-(rz-px-qy)rz\}$

$$=\lambda\{p^2x^2-q^2y^2-r^2z^2+2qryz\}$$
$$=\lambda(px+qy-rz)(px-qy+rz)$$
$$=\frac{1}{\lambda}\cdot\frac{bc}{qr}=\frac{1}{\lambda}\cdot\frac{abc}{pqr}\cdot\frac{p}{a}.$$

Hence $a(ax-by-cz)/p=b(by-cz-ax)/q=c(cz-ax-by)/r$.

16. If $ab=cd$, then

$$(a+c)(a+d)(b+c)(b+d)/(a+b+c+d)^2$$
$$=\frac{(a+c)(a+ab/c)(b+c)(b+ab/c)}{(a+b+c+ab/c)^2}$$
$$=\frac{(a+c)(ac+ab)(b+c)(bc+ab)}{(ac+bc+c^2+ab)^2}$$
$$=\frac{ab(a+c)^2(b+c)^2}{\{(a+c)(b+c)\}^2}=ab.$$

Again, if $a+b=c+d$,

$$abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)/(ab+cd)$$

$$=\frac{(bc+ca+ab)(a+b-c)+abc}{ab+c(a+b-c)}=\frac{(bc+ca+ab)(a+b)-c^2(a+b)}{bc+ca+ab-c^2}=a+b.$$

17. (i) $$L_{x=2}\frac{x^2-7x+10}{x^2-9x+14}=L_{x=2}\frac{(x-2)(x-5)}{(x-2)(x-7)}=L_{x=2}\frac{x-5}{x-7}=\frac{3}{5},$$

$$L_{x=\infty}\frac{x^2-7x+10}{x^2-9x+14}=L_{x=\infty}\frac{x^2\left(1-\frac{7}{x}+\frac{10}{x^2}\right)}{x^2\left(1-\frac{9}{x}+\frac{14}{x^2}\right)}=L_{x=\infty}\frac{x^2}{x^2}=1.$$

(ii) $$L_{x=2}\frac{x^2-4x+4}{x^2-5x+6}=L_{x=2}\frac{(x-2)(x-2)}{(x-2)(x-3)}=L_{x=2}\frac{x-2}{x-3}=0,$$

$$L_{x=\infty}\frac{x^2-4x+4}{x^2-5x+6}=L_{x=\infty}\frac{x^2\left(1-\frac{4}{x}+\frac{4}{x^2}\right)}{x^2\left(1-\frac{5}{x}+\frac{6}{x^2}\right)}=1.$$

(iii) $$L_{x=2}\frac{x^2+6x-16}{x^3-12x+16}=L_{x=2}\frac{(x-2)(x+8)}{(x-2)(x^2+2x-8)}=L_{x=2}\frac{x+8}{x^2+2x-8}$$
$$=L_{x=2}\frac{10}{0}=\infty,$$

$$L_{x=\infty}\frac{x^2+6x-16}{x^3-12x+16}=L_{x=\infty}\frac{x^2\left(1+\frac{6}{x}-\frac{16}{x^2}\right)}{x^3\left(1-\frac{12}{x^2}+\frac{16}{x^3}\right)}=L_{x=\infty}\frac{x^2}{x^3}=L_{x=\infty}\frac{1}{x}$$
$$=0.$$

XXI. Pages 267—270.

1. $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ will be in A. P. provided
$$a^2(b+c)-b^2(c+a)=b^2(c+a)-c^2(a+b),$$
that is
$$(a-b)(bc+ca+ab)=(b-c)(bc+ca+ab),$$
and this condition is satisfied since $a-b=b-c$.

2. The numbers can be taken to be $a-3b$, $a-b$, $a+b$ and $a+3b$. Then $4a^2+20b^2=120$ and $a^2-b^2-(a^2-9b^2)=8$, whence $b=\pm1$ and $a=\pm5$. Thus the numbers are **2, 4, 6, 8** and **−8, −6, −4, −2.**

3. We have (i) $2b=a+c$ and (ii) $\frac{2}{c}=\frac{1}{b}+\frac{1}{d}$. From (ii) $2bd=(b+d)c$, and hence from (i) $(a+c)d=(b+d)c$; $\therefore\ ad=bc$.

4. Let the numbers be a, ar, ar^2.
$$a(1+r+r^2)=14 \text{ and } a^2(1+r^2+r^4)=84;$$
hence by division $a(1-r+r^2)=6$.
$$\therefore\ (1+r+r^2)/(1-r+r^2)=14/6,$$
whence $r=2$ or $\frac{1}{2}$; and then $a=2$ or 8.

Thus the numbers are **2, 4, 8.**

5. We have $2b=a+c$ and therefore $2b^2=ab+bc$. But $x^2=ab$ and $y^2=bc$; $\therefore\ 2b^2=x^2+y^2$.

6. $\frac{a}{b+c-a}$, $\frac{b}{c+a-b}$ and $\frac{c}{a+b-c}$ will be in H. P. if
$$\frac{b+c-a}{a}+\frac{a+b-c}{c}-\frac{2(c+a-b)}{b}=0,$$
or if
$$\frac{b+c-a}{a}+2+\frac{a+b-c}{c}+2-\frac{2(c+a-b)}{b}-4=0,$$
that is if
$$(a+b+c)\left(\frac{1}{a}+\frac{1}{c}-\frac{2}{b}\right)=0;$$
and this condition is satisfied since a, b, c are in H. P.

7. We have $\frac{1}{a}-\frac{1}{b}=\frac{1}{b}-\frac{1}{c}=\frac{1}{c}-\frac{1}{d}$;

therefore $\frac{1}{a}-\frac{1}{d}=\left(\frac{1}{a}-\frac{1}{b}\right)+\left(\frac{1}{b}-\frac{1}{c}\right)+\left(\frac{1}{c}-\frac{1}{d}\right)=3\left(\frac{1}{a}-\frac{1}{b}\right)$,

or $\frac{d-a}{ad}=3\,\frac{b-a}{ab}$. Also $\frac{c-b}{bc}=\frac{d-c}{cd}$.

Hence $\frac{(d-a)(c-b)}{abcd}=3\,\frac{(b-a)(d-c)}{abcd}$.

8. $\frac{2}{b}=\frac{1}{b-a}+\frac{1}{b-c}$, if $2(b-a)(b-c)=b(2b-a-c)$,

or if $2ac-b(a+c)=0$, or $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$;

and this is known.

9. $\frac{b+a}{b-a}+\frac{b+c}{b-c}=2$ if $2b^2-2ac=2b^2-2(a+c)b+2ac$;

that is, if $(a+c)b=2ac$, which is the case since a, b, c are in H. P.

10. We have (i) $2b=a+c$, (ii) $c^2=bd$ and (iii) $\frac{2}{d}=\frac{1}{c}+\frac{1}{e}$.

Hence $2c^2=2bd=(a+c)\times\frac{2ce}{c+e}$;

therefore $c^2(c+e)=ce(c+a)$, whence $c^2=ae$.

11. $a-\frac{b}{2}$, $\frac{b}{2}$, $c-\frac{b}{2}$ will be in G. P. provided

$$b^2=(2a-b)(2c-b),\text{ or }2ac-b(a+c)=0,$$

and this is the case since a, b, c are in H. P.

12. a, $a-c$, $a-b$ will be in H. P. if $\frac{1}{a}-\frac{1}{a-c}=\frac{1}{a-c}-\frac{1}{a-b}$,

or $\frac{-c}{a(a-c)}=\frac{c-b}{(a-c)(a-b)}$, or $c(a-b)=a(b-c)$,

or $\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$, which is known. The second relation is the same as the first if the H. P. be read backwards.

13. We have (i) $a_1+a_2=x+y$, (ii) $g_1g_2=xy$ and (iii) $\frac{1}{h_1}+\frac{1}{h_2}=\frac{1}{x}+\frac{1}{y}$.

Hence $\frac{h_1+h_2}{h_1h_2}=\frac{x+y}{xy}=\frac{a_1+a_2}{g_1g_2}$.

14. Let a, ar, ar^2,... be the G. P.; then we have

$$4a(1+r+r^2)=a(r^2+r^3+r^4),$$

whence $r^2=4$ and therefore $r=\pm2$. Also $ar^6=384$, whence $a=6$. Thus the series is 6, ±12, 24, &c.

15. We have $\frac{1}{a_1}-\frac{1}{a_2}=\frac{1}{a_2}-\frac{1}{a_3}=\ldots=\frac{1}{a_{n-1}}-\frac{1}{a_n}=\lambda$ suppose.

Hence $\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+\ldots+\left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)=(n-1)\lambda,$

that is $$\frac{1}{a_1}-\frac{1}{a_n}=(n-1)\lambda \quad \text{(i)}.$$

Also $a_2-a_1=a_1a_2\lambda,\ a_3-a_2=a_2a_3\lambda,\ldots,\ a_n-a_{n-1}=a_{n-1}a_n\lambda;$

therefore $$a_n-a_1=(a_1a_2+\ldots+a_{n-1}a_n)\lambda \quad \text{(ii)}.$$

From (i) and (ii) $(a_1a_2+\ldots+a_{n-1}a_n)\lambda=(n-1)\,a_1a_n\lambda.$

16. As in Art. 222, $x=\frac{1}{3}(2a+b),\ y=\frac{1}{3}(a+2b).$

And, as in Art. 233, $u=\frac{3ab}{a+2b},\ v=\frac{3ab}{2a+b}.$

Hence $vx=uy=ab.$

17. Let the numbers be $a-b,\ a,\ a+b.$ Then we have (i) $a^2-b^2=5a$ and (ii) $2a+b=8\,(a-b).$ Whence the numbers are found to be 3, 9, 15.

18. We have $$\frac{a+b}{1-ab}-b+\frac{b+c}{1-bc}-b=0;$$

$$\therefore\ \frac{a\,(1+b^2)}{1-ab}+\frac{c\,(1+b^2)}{1-bc}=0;$$

$$\therefore\ \frac{a}{1-ab}+\frac{c}{1-bc}=0,$$

since $$b^2+1\neq 0;$$

$$\therefore\ 2abc=a+c, \text{ or } 2b=\frac{1}{a}+\frac{1}{c},$$

whence $a,\ \frac{1}{b}$ and c are in H. P.

19. We have $a+c=2b$ and $\frac{2}{b^2}=\frac{1}{a^2}+\frac{1}{c^2}.$

Hence $2a^2c^2=b^2\,(a^2+c^2)=b^2\,(4b^2-2ac);\ \therefore\ (2b^2+ac)\,(b^2-ac)=0.$

Hence $2b^2+ac=0$, shewing that $-\frac{a}{2},\ b,\ c$ are in G. P.; or else $b^2-ac=0$ from which it follows that $a=b=c.$

20. Let $x,\ y$ be the nth terms in the two series; then

$$x=a+(n-1)\,(b-a) \text{ and } \frac{1}{y}=\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right).$$

Hence $$y=\frac{ab}{b+(n-1)\,(a-b)}=\frac{ab}{a+b-x};$$

therefore $$y\,(a-x)=b\,(a-y).$$

21. We have $b+c=2a$ and $b^2=ac$; hence

$$\frac{1}{b}+\frac{c}{b^2}=\frac{2}{c}, \text{ that is } \frac{1}{b}+\frac{1}{a}=\frac{2}{c}.$$

22. The last term in the kth group is $1+2+3+\ldots+k=\frac{1}{2}k(k+1)$, and the last term in the $(k-1)$th group is $\frac{1}{2}k(k-1)$. Hence we have to find the sum of k numbers in A. P. of which the first is $\frac{1}{2}k(k-1)+1$ and the last $\frac{1}{2}k(k+1)$; the sum is therefore $\frac{k}{2}\left\{\frac{1}{2}k(k+1)+\frac{1}{2}k(k-1)+1\right\}=\frac{1}{2}k(k^2+1)$.

23. As in Art. 222, the $(r+1)$th term of the A. P. $=\dfrac{lr+a(n-r-1)}{n-1}$;

and, as in Art. 233, the $(n-r)$th term of the H. P. $=\dfrac{(n-1)\,al}{lr+a(n-r-1)}$.

24. Let a be the given term, and d the common difference; then the different products are $(a-d)(a+d)$, $(a-2d)(a+2d)$, &c. Hence the differences are $3d^2$, $5d^2$, $7d^2$, &c.

25. Let $a, ar, \ldots$ be the G. P.; then $S_n=\dfrac{a(1-r^n)}{1-r}$,

$$S_{2n}=\frac{a(1-r^{2n})}{1-r} \text{ and } S_{3n}=\frac{a(1-r^{3n})}{1-r}.$$

Hence $S_n(S_{3n}-S_{2n})=a^2\dfrac{1-r^n}{(1-r)^2}(r^{2n}-r^{3n})=\dfrac{a^2r^{2n}(1-r^n)^2}{(1-r)^2}=(S_{2n}-S_n)^2$.

26. First let a, b, c be in A. P. and let $a>c$ and therefore $a>b>c$. Then, since $a>b>c$,

$$a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}>b^{n-1}+b^{n-2}c+b^{n-3}c^2+\ldots+c^{n-1}.$$

But $a-b=b-c$;

$$\therefore\ (a-b)(a^{n-1}+a^{n-2}b+\ldots+b^{n-1})>(b-c)(b^{n-1}+b^{n-2}c+\ldots+c^{n-1}),$$

that is $$a^n-b^n>b^n-c^n.$$

Next, let a, b, c be in G. P. Let $a>c$; then, since $b^2=ac$, $a>b>c$.

Then $(a^n-c^n)^2>0$; $\therefore\ (a^n+c^n)^2>4a^nc^n$, that is $(a^n+c^n)^2>4b^{2n}$ and therefore $a^n+c^n>2b^n$.

Lastly, let a, b, c be in H. P., and let x be the Arithmetical Mean of a and c.

Then we know [Art. 228] that $x>b$; and, by the first case, $a^n+c^n>2x^n$. Hence $a^n+c^n>2b^n$.

27. (i) Let a, $a+d$, &c., be the series; then

$$P=a+(p-1)\,d,\ Q=a+(q-1)d,\ R=a+(r-1)\,d.$$

Hence $P\,(q-r)+\ldots=a\{q-r+r-p+p-q\}$

$$+d\,\{(p-1)\,(q-r)+(q-1)\,(r-p)+(r-1)\,(p-q)\}=0.$$

(ii) Let a, af, &c. be the series; then $P=af^{p-1}$, &c.; therefore

$$P^{q-r}\,.\,Q^{r-p}\,.\,R^{p-q}=\{af^{p-1}\}^{q-r}\,.\,\{af^{q-1}\}^{r-p}\,.\,\{af^{r-1}\}^{p-q}$$
$$=a^{q-r+r-p+p-q}\,f^{(p-1)(q-r)+\cdots}=a^0\,.\,f^0=1.$$

(iii) Let a be the first term of the series, and d the common difference of the corresponding arithmetical progression; then we have

$$\frac{1}{P}=\frac{1}{a}+(p-1)\,d,\ \&c.$$

Hence $\dfrac{q-r}{P}+\dfrac{r-p}{Q}+\dfrac{p-q}{R}=\dfrac{1}{a}(q-r+r-p+p-q)$

$$+d\,\{(p-1)\,(q-r)+(q-1)\,(r-p)+(r-1)\,(p-q)\}=0;$$

$$\therefore\ QR\,(q-r)+\ldots=0.$$

28. Since a_1, a_2, a_3, &c. are in H.P., $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}$, &c. are in A.P.; therefore

$\dfrac{a_1+a_2+\ldots+a_n}{a_1}$, $\dfrac{a_1+a_2+a_3+\ldots+a_n}{a_2}$, $\dfrac{a_1+a_2+a_3+\ldots+a_n}{a_3}$, &c. are in A.P.,

and hence also $\dfrac{a_1+a_2+\ldots+a_n}{a_1}-1$, $\dfrac{a_1+a_2+\ldots+a_n}{a_n}-1$, &c. are in A.P.,

that is $\dfrac{a_2+a_3+\ldots+a_n}{a_1}$, $\dfrac{a_1+a_3+\ldots+a_n}{a_2}$, &c. are in A.P.

And since $\dfrac{a_2+a_3+\ldots+a_n}{a_1}$, $\dfrac{a_1+a_3+\ldots+a_n}{a_2}$, &c. are in A.P., their reciprocals are in H.P.

29. We may write

$$(a_1^2+a_2^2+a_3^2+\ldots+a_{n-1}^2)\,(a_2^2+a_3^2+\ldots+a_n^2)-(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)^2=0$$

in the form

$$[(a_1a_3-a_2^2)^2+(a_1a_4-a_2a_3)^2+(a_1a_5-a_2a_4)^2+\ldots+(a_1a_n-a_2a_{n-1})^2]$$
$$+[(a_2a_4-a_3^2)^2+(a_2a_5-a_3a_4)^2+\ldots+(a_2a_n-a_3a_{n-1})^2]$$
$$+[(a_3a_5-a_4^2)^2+(a_3a_6-a_4a_5)^2+\ldots+(a_3a_n-a_4a_{n-1})^2]$$
$$+\ldots[(a_{n-3}a_{n-1}-a_{n-2}^2)^2+(a_{n-3}a_n-a_{n-2}a_{n-1})^2]+(a_{n-2}a_n-a_{n-1}^2)^2=0.$$

Hence, as all the quantities are real,

$$a_1a_3-a_2^2=0,\ a_2a_4-a_3^2=0,\ldots a_{n-2}a_n-a_{n-1}^2=0,$$

and therefore $\dfrac{a_1}{a_2}=\dfrac{a_2}{a_3}=\dfrac{a_3}{a_4}=\ldots=\dfrac{a_{n-1}}{a_n}$.

30. Put in succession $n=1$ and $n=2$; then $S_1=4$ and $S_2=16$; thus, if the theorem be true, the series must be $4+12+\ldots$, and it is easily seen that the sum of n terms of the A.P. $4+12+\ldots$ is $4n^2$.

In the second case we have $S_1+1=9$ and $S_2+1=25$. Hence, if the theorem be true, the series must be $8+16+\ldots$; and it is easily seen that the sum of n terms of the A.P. $8+16+\ldots$ is $(2n+1)^2-1$.

31. The sum of p terms of 1, 3, 5, 7,... beginning at $2k+1$ is easily seen to be equal to $p(p+2k)$. Hence $p(p+2k)=p^r$ and therefore $k=\frac{1}{2}(p^{r-1}-p)$.

Now, if $r>1$, p^{r-1} is even or odd according as p is even or odd; hence $p^{r-1}-p$ is always *even*, and therefore k is as it should be an *integer*.

32. Let a, b be the given first and second terms; then the $(r+1)$th terms of the two series are respectively $a+r(b-a)$ and $a\left(\frac{b}{a}\right)^r$.

We have to shew that

$$a\left(\frac{b}{a}\right)^r-\{a+r(b-a)\}>0,$$

or
$$b^r-a^r-r(b-a)a^{r-1}>0,$$

that is
$$(b-a)\{b^{r-1}+ab^{r-2}+a^2b^{r-3}+\ldots+a^{r-1}-ra^{r-1}\}>0,$$

or
$$(b-a)\{(b^{r-1}-a^{r-1})+a(b^{r-2}-a^{r-2})+\ldots+a^{r-2}(b-a)>0.$$

Now $(b-a)(b^s-a^s)$ is always positive for both factors are positive if $b>a$ and both are negative if $b<a$.

XXII. Pages 277—279.

1. Let a, b be the digits; then $7a+b=9b+a$; $\therefore 3a=4b$. Hence, as a and b must both be less than 7, $a=4$ and $b=3$.

2. The greatest number will obviously be that which has $r-1$ for each of its digits, and the smallest will have all its digits zero except the first on the left which must be unity.

3. Let a, b, c be the digits; then the number is

$$a\,.\,10^5+b\,.\,10^4+c\,.\,10^3+a\,.\,10^2+b\,.\,10+c=1001\,(100a+10b+c).$$

4. Expressed in the scale of 2 the number 1027 is 10000000011. Hence $1027=2^{10}+2+1$, so that the weights required are 1 lb., 2 lbs., and 1024 lbs.

5. The number is $r^2+4r+4=(r+2)^2$.

6. The numbers are respectively

$$r^2+2r+1,\ r^4+2r^3+3r^2+2r+1 \text{ and } r^6+2r^5+3r^4+4r^3+3r^2+2r+1,$$

that is
$$(r+1)^2,\ (r^2+r+1)^2 \text{ and } (r^3+r^2+r+1)^2.$$

7. Let a, b be the two digits ; then we have

$$10a+b+18=10b+a, \text{ and } 10a+b=7b+a,$$

whence $$a=4 \text{ and } b=6.$$

8. In the scale of 5, as in Art. 234, Ex. 4,

$$\cdot 4\dot{4}\dot{0}=\frac{440-4}{440}=\frac{431}{440} \text{ which in the ordinary scale } =\frac{116}{120}=\frac{29}{30}.$$

And, in the scale r, $\cdot 5\dot{4}\dot{4}=\dfrac{5r+4-5}{(r-1)\,r}$.

Hence $$\frac{29}{30}=\frac{5r-1}{r^2-r}, \text{ or } 29r^2-179r+30=0,$$

that is $$(r-6)\,(29r-5)=0.$$

Hence $r=6$, the value $r=\dfrac{5}{29}$ being inadmissible.

9. By Art. 235, $N-S$ and $2N-2Q$ are both multiples of 9, and therefore also $2Q\sim 2S$ is a multiple of 9. Hence, as 2 is prime to 9, $Q\sim S$ must be a multiple of 9.

10. Since $N-S$ is divisible by $r-1$, and $r-1$ is even, it follows that $N-S$ must be even, and therefore N and S must be both even or both odd.

11. and **12.** Let the original number be $p_0r^n+p_1r^{n-1}+\ldots+p_n$; then we have to shew that

$$(p_0r^n+p_1r^{n-1}+\ldots+p_{n-1}r+p_n)^2-(p_nr^n+p_{n-1}r^{n-1}+\ldots+p_1r+p_0)^2,$$

that is

$$\{(r^n+1)\,(p_0+p_n)+r\,(r^{n-2}+1)\,(p_1+p_{n-1})+r^2\,(r^{n-4}+1)\,(p_2+p_{n-2})+\ldots\}$$
$$\times\{(p_0-p_n)\,(r^n-1)+r\,(r^{n-2}-1)\,(p_1-p_{n-1})+r^2\,(r^{n-4}-1)\,(p_2-p_{n-2})+\ldots\},$$

is a multiple of r^2-1.

Now, if n be odd, r^n+1, $r^{n-2}+1$, &c., are all divisible by $r+1$, and therefore the first bracket is divisible by $r+1$; also r^n-1, $r^{n-2}-1$, &c., are all divisible by $r-1$, and therefore the second bracket is divisible by $r-1$. Hence the whole product is divisible by r^2-1.

Again, if n be even, r^n-1, $r^{n-2}-1$, &c. are all divisible by r^2-1.

Hence the proposition is true for all values of n.

13. We have $$7^2x+7y+z=11^2z+11y+x,$$

that is $$48x-4y-120z=0, \text{ or } 12x-y-30z=0.$$

Now, since $12x$ and $30z$ are both divisible by 6, y must be divisible by 6. Hence, as y cannot be greater than 6, we must have either $y=6$ or $y=0$. If $y=6$, we have $2x-1=5z$, and therefore $2x-1$ must be a multiple of 5. Hence, as x is not greater than 6, the only possible value is $x=3$, giving $z=1$, so that the number in this case is 361.

If $y=0$, we have $2x=5z$; hence, as x is not greater than 6, we must have $x=5$ and $z=2$, so that the number in this case is 502.

14. Since any number in the scale of 5 when divided by 4 will leave the same remainder as when the sum of the digits is divided by 4, it follows that all the possible numbers which have the same digits will leave the same remainder.

Again, the difference of any number in the scale of 9 and the sum of its digits is divisible by 8, and therefore by 4. Hence all possible numbers in the scale of 9 which have the same digits will leave the same remainder when divided by 4.

15. We have

$$p_5 \cdot 3^5 + p_4 \cdot 3^4 + p_3 \cdot 3^3 + p_2 \cdot 3^2 + p_1 \cdot 3 + p_0 = p_2 \cdot 12^2 + p_1 \cdot 12 + p_0,$$

each of the letters being 0, 1 or 2 but p_5 and p_2 are by supposition not zero.

Cancel p_0, and divide by 3; then

$$p_5 \cdot 3^4 + p_4 \cdot 3^3 + p_3 \cdot 3^2 = 45p_2 + 3p_1.$$

Now in the above all the terms are divisible by 9 except $3p_1$; hence, as $p_1 < 3$, p_1 must be zero. We then have

$$9p_5 + 3p_4 + p_3 = 5p_2.$$

Now p_2 must be 1 or 2; and, if p_2 were 1, p_5 would have to be zero otherwise $9p_5$ would of itself be greater than $5p_2$. Hence $p_2=2$ and the number is therefore $2 \cdot 12^2 + p_0$, that is 288, 289 or 290 since p_0 can be 0, 1 or 2.

16. Let a, b, c, d be the digits, then we have

$$\begin{array}{r} abcd \\ \underline{2} \\ dcba \end{array}$$

It is obvious that d is greater than a, we must therefore have $2d=a+8$. Then $2c+1=b$ or $2c+1=b+8$.

If $2c+1=b$, we have $2b=c$ or $2b=c+8$, in the first case $2a=d$ which is inconsistent with $2d=a+8$, and in the second case $2a+1=d$ which with $2d=a+8$ gives $a=2$ and $d=5$. Also $2c+1=b$ and $2b=c+8$ give $c=2$ and $b=5$, so that we have the number 2525.

Again, if $2c+1=b+8$, we have also $2b+1=c$ or $2b+1=c+8$; in the first case $2a=d$ which is inconsistent with $2d=a+8$, and in the second case $2a=d+1$ which with $2d=a+8$ gives $a=2$ and $d=5$. Also $2c+1=b+8$ and $2b+1=c+8$, give $b=c=7$, so that we have the number 2775.

Hence the only two numbers which satisfy the conditions are 2525 and 2775.

17. Let the digits be $a+2x$, $a+x$, a; then we have

$$100(a+2x)+10(a+x)+a=15(3a+3x).$$

But, since the number is divisible by 15, and therefore by 5, the last digit must be zero or 5; and the last digit is not zero for the number is increased by inverting the order of the digits. Hence $a=5$, and we have

$$100(2x+5)+10(x+5)+5=45(x+5),$$

whence $x=-2$, and therefore the number is 135.

18. $792=8\times9\times11$. Now, since 1000 is a multiple of 8, the number $45c$ must be divisible by 8, and it is seen at once that 456 is the only number between 450 and 460 which is divisible by 8. Hence $c=6$, and from Art. 235

we have $$a+b+6+13=\text{a multiple of } 9,$$

and $$1+a+4+6\sim(3+b+5)=\text{a multiple of } 11.$$

Hence $a-b+3=11\,.\,x$, where x is an integer; and, since a and b are both positive and less than 10, it follows that x is 0 or 1.

If $x=0$, $a-b+3=0$, and therefore $2a+4=9y$; and, as a must be an integer less than 10, y must be 2 and $a=7$. But, if $a=7$, $b=a+3=10$, which is inadmissible.

If $x=1$, $a-b+3=11$, and therefore $2a+2=$ a multiple of 9, whence $a=8$, and then $8-b+3=11$ gives $b=0$. Hence the only number which satisfies the condition is 1380456.

19. $$r^3+r^2+5r+5=(r+2)(r^2-r+7)-9.$$

Hence r^3+r^2+5r+5 cannot be divisible by $r+2$ unless 9 is divisible by $r+2$, whence it follows that $r+2=9$ or $r+2=3$, so that the only possible values of r are 7 and 1, the latter value being inadmissible.

20. We have $\dfrac{abcd}{\quad 9}$.
$\overline{dcba}$

Since $a\nless1$, $d\ngtr9$ and $9a\ngtr d$, it follows that $a=1$ and $d=9$. Hence we have $9c+8=b+10x$, and $9b+x=c$; whence

$$81b+9x+8=b+10x, \text{ or } 80b+8=x;$$

$\therefore$ $b=0$ and $x=8$, since $x\ngtr8$. And since $b=0$ and $x=8$, we have $c=8$. Thus the number must be 1089.

21. $$(r^2-1)(r^n-1)=r^{n+2}-r^n-r^2+1$$

$$=(r-1)r^{n+1}+(r-2)r^n+(r-1)r^{n-1}+(r-1)r^{n-2}+\ldots+(r-1)r^3+(r-1)r^2+0\,.\,r+1.$$

Hence when $(r^2-1)(r^n-1)$ is expressed in the scale of r, the digits in order from left to right are

$$r-1,\ r-2,\ r-1,\ r-1,\ \ldots\ldots,\ r-1,\ 0,\ 1.$$

Again $$(r+1)(r^n-1)=r^{n+1}+r^n-r-1$$

$$=r^{n+1}+0\,.\,r^n+(r-1)r^{n-1}+(r-1)r^{n-2}+\ldots+(r-1)r^2+(r-2)r+(r-1).$$

Hence when $(r+1)(r^n-1)$ is expressed in the scale of r, the digits in order from left to right are

$$1,\ 0,\ r-1,\ r-1, r-1,\ \ldots\ldots,\ r-1,\ r-2,\ r-1.$$

22. Let $F=\cdot\dot{0}123\ldots\ldots(r-4)\,(r-3)\,(r-2)\,(r-1)\,;$

then $r\,.\,F=\cdot\dot{1}234\ldots\ldots(r-3)\,(r-1)\,\dot{0}.$

Hence $(r-1)\,F=\cdot 1111\ldots\ldots$

$$=\frac{1}{r-1};$$

$$\therefore\ F=\frac{1}{(r-1)^2}.$$

23. Let $N=abcdef$; then by supposition $3N=bcdefa$. But

$$10N=bcdefa+a\,(10^6-1).$$

Hence $$7N=a\,(10^6-1).$$

We have also the following relations

$$3f=a+10x,\ \ x+3e=f+10y,\ \ y+3d=e+10z,\ \ z+3c=d+10u,$$

$$u+3b=c+10v \text{ and } v+3a=b.$$

The last relation shews that a cannot be greater than 3, and if $a=3$ b must be 9 and $v=0$. But $b=9$, $v=0$ are inconsistent with $u+3b=c+10v$. Hence a must be 1 or 2.

If $$a=1,\ N=\frac{1}{7}(10^6-1)=142857;$$

and if $$a=2,\ N=\frac{2}{7}(10^6-1)=285714.$$

24. We have $xyy\times z=yyz$. Hence $yz=z+10m$, that is $(y-1)\,z=10m$.

Hence either $y-1$ or z must be divisible by 5, but $y-1$ and z are both less than 10, and therefore either $y-1=5$ or $z=5$.

If $y=6$, we have $x66\times z=66z$, hence z must be even and by trial the only value of z is 4, the corresponding value of x being 1.

If $z=5$, we have $xyy\times 5=yy5$, hence y must be odd and not less than 5, and by trial the only possible value of y is 9, the corresponding value of x being 1. Thus the only numbers are 166 and 199.

There is one other case, namely when $m=0$, and then $y=z=x=1$.

XXIII. Pages 294—296.

1. The number of different sets of four which the first can have is ${}_{20}C_4$; and for every one of these the second can have ${}_{16}C_4$ different sets of four; and the third can now have ${}_{12}C_4$ different sets; &c.

Hence the number required is

$$\frac{\lfloor 20}{\lfloor 4 \lfloor 16} \times \frac{\lfloor 16}{\lfloor 4 \lfloor 12} \times \frac{\lfloor 12}{\lfloor 4 \lfloor 8} \times \frac{\lfloor 8}{\lfloor 4 \lfloor 4} = \frac{\lfloor 20}{(\lfloor 4)^5}.$$

2. If all the four men who can only row on the bow side are taken there are ${}_7C_4 = 35$ ways of choosing the four men for the stroke side.

If three of the four men who can only row on the bow side are taken, one or other of the double men must be taken for the bow side, and there are ${}_5C_4$ ways of choosing four men for the stroke side without choosing the other double man, and ${}_5C_3$ ways of making up the stroke side including the double man. The whole number in this case is therefore

$${}_4C_3 \cdot {}_2C_1 \cdot {}_5C_4 + {}_4C_3 \cdot {}_5C_3 = 80.$$

If only two of the four men who can only row bow are taken, both the double men must be put on the bow side, and there are ${}_5C_4$ ways of choosing four men for the stroke side. The whole number in this case is therefore

$${}_4C_2 \cdot {}_5C_4 = 30.$$

The number required $= 35 + 120 + 30 = 185.$

3. Each elector can vote in three ways and therefore the whole number of ways is 3^{12}.

The number of ways in which A can receive four votes is ${}_{12}C_4$; and if any particular set of four vote for A, the number of ways in which four of the remaining electors can vote for B is ${}_8C_4$. Hence the total number of ways in which each candidate can receive four votes is

$$\frac{\lfloor 12}{\lfloor 4 \lfloor 8} \cdot \frac{\lfloor 8}{\lfloor 4 \lfloor 4} = \frac{\lfloor 12}{\{\lfloor 4\}^3}.$$

4. $${}_{4n}C_{2n} = \frac{\lfloor 4n}{\lfloor 2n \lfloor 2n} \text{ and } {}_{2n}C_n = \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

Now $\lfloor 4n = \{1.3.5\ldots(4n-1)\}\{2.4.6\ldots 4n\} = \{1.3.5\ldots(4n-1)\}\, 2^{2n} \lfloor 2n,$

and $\lfloor 2n = \{1.3.5\ldots(2n-1)\}\{2.4.6\ldots 2n\} = \{1.3.5\ldots(2n-1)\}\, 2^n \lfloor n.$

Hence ${}_{4n}C_{2n} : {}_{2n}C_n$

$= 2^{2n} \lfloor 2n \{1.3.5\ldots(4n-1)\} . \lfloor n . \lfloor n : \{1.3.5\ldots(2n-1)\}^2 2^{2n} . \lfloor n . \lfloor n . \lfloor 2n$

$= \{1.3.5\ldots(4n-1)\} : \{1.3.5\ldots(2n-1)\}^2.$

5. If all the ten digits are used we have ${}_{10}P_{10} - {}_9P_9$ different numbers, since using 0 as the *first* figure is the same as using only nine. Similarly, when nine of the digits are used there are ${}_{10}P_9 - {}_9P_8$ different numbers; and so on. Hence the number required

$= ({}_{10}P_{10} - {}_9P_9) + ({}_{10}P_9 - {}_9P_8) + ({}_{10}P_8 - {}_9P_7) + \ldots\ldots + ({}_{10}P_2 - {}_9P_1) + {}_9P_1$

$= 3265920 + 3265920 + 1632960 + 544320 + 136080 + 27216 + 4536 + 648 + 81 + 9 = 8877690.$

6. If p particular things occur there is only a choice of $r-p$ things out of the remaining $n-p$ things, so that there are ${}_{n-p}C_{r-p}$ sets of things satisfying the conditions; but each set can be permuted in $\lfloor r$ different ways. Hence the whole number of permutations

$$= \lfloor r \cdot \frac{\lfloor n-p}{\lfloor n-r \lfloor r-p} = \frac{\lfloor r}{\lfloor r-p} \cdot \frac{\lfloor n-p}{\lfloor n-r} = {}_rP_p \cdot {}_{n-p}P_{r-p}.$$

7. The number of straight lines is equal to the number of ways of taking the n points two together $= \frac{1}{2} n(n-1)$.

8. If no three of the n points were on a straight line the number of lines would be $\frac{1}{2} n(n-1)$. But if m points are on a straight line, there is only one straight line formed by joining these m points in pairs instead of $\frac{1}{2} m(m-1)$. Hence the number required is $\frac{1}{2} n(n-1) - \left\{\frac{1}{2} m(m-1) - 1\right\}$.

9. If no three of the n points were on a straight line there would be $\frac{n(n-1)(n-2)}{\lfloor 3}$ triangles formed by taking the n points in threes. But, if m points are on a straight line, the above number of triangles would be diminished by the number of triangles which could be formed from these m points on the supposition that no three of them were on a straight line. The required number of triangles is therefore

$$\frac{n(n-1)(n-2)}{\lfloor 3} - \frac{m(m-1)(m-2)}{\lfloor 3}.$$

10. Starting from any one of the points we can go to any one of the $n-1$ remaining points, and thence to any one of the remaining $n-2$ points, and so on to the place of starting. The number of different ways of describing a polygon is therefore $\lfloor n-1$. But the polygon $ABC...XYZA$ is the same polygon as $AZYX...CBA$ although described in the reverse order; and hence the number of different polygons is $\frac{1}{2} \lfloor n-1$.

11. The number of different lines joining the points is $\frac{1}{2} n(n-1)$. Also the line joining any two of the points is cut by all the lines through the remaining $n-2$ points, and therefore on each line there are $\frac{1}{2}(n-2)(n-3)$ of the required intersections; hence, as each intersection is on two lines, the whole number of intersections required

$$= \frac{1}{2} \cdot \frac{1}{2}(n-2)(n-3) \cdot \frac{1}{2} n(n-1).$$

12. Each of the m lines through one angular point of the triangle is cut by all the $2m$ lines through the other two angular points. There are therefore $2m$ points of intersection on each of the $3m$ lines; hence, as each point of intersection is on two lines, the whole number of points of intersection is

$$\frac{1}{2} \cdot 2m \cdot 3m.$$

13. The man must pass over $m+n-2$ portions of straight road, $m-1$ parts going W. to E. and $n-1$ parts N. to S. The number of different paths is therefore equal to the number of ways in which $m-1$ things can be chosen out of $m+n-2$,

that is $$\frac{\lfloor m+n-2}{\lfloor m-1 \lfloor n-1}.$$

14. This is the same as the second case of Ex. 4, Art. 251.

15. There are $\frac{\lfloor 2n}{\lfloor n \lfloor n}$ ways of choosing n persons to sit at the first table. Also the number of different ways in which n persons can be arranged at a round table is $\lfloor n-1$, for one person can be supposed to be fixed and the number of ways the others can be arranged is ${}_{n-1}P_{n-1}$. Hence the total number of ways required

$$= \frac{\lfloor 2n}{\lfloor n \lfloor n} \times \lfloor n-1 \times \lfloor n-1 = \frac{\lfloor 2n}{n^2}.$$

16. There are $m+2$ lines in each direction, and a parallelogram is bounded by any two lines in one direction and by any two in the other direction. The total number of parallelograms is therefore

$$\frac{1}{2}(m+2)(m+1) \times \frac{1}{2}(m+2)(m+1).$$

17. Suppose the p positive signs to be written down, then, counting the outside places, there are $p+1$ places in which the n negative signs can be put only one being put in each place. The number of ways in which this can be done is

$${}_{p+1}C_n = \frac{\lfloor p+1}{\lfloor n \lfloor p+1-n}.$$

18. This is clearly the same as the number of homogeneous products of m dimensions which can be formed with $n+1$ things; and

$${}_{n+1}H_m = \frac{(n+1)(n+2)\ldots(n+m)}{1 \cdot 2 \ldots m} = \frac{\lfloor m+n}{\lfloor n \lfloor m}.$$

19. The first pair can be taken in ${}_{2n}C_2$ ways; then the next in ${}_{2n-2}C_2$ ways; and so on. Hence, as the order in which the pairs are taken is immaterial, the required number

$$= {}_{2n}C_2 \cdot {}_{2n-2}C_2 \ldots {}_4C_2 \cdot {}_2C_2 \div \lfloor n.$$

20. The first set of n things can be chosen in ${}_{mn}C_n$ ways; then the next n can be chosen in ${}_{(m-1)n}C_n$ ways; and so on. Hence, as the order in which the sets are taken is immaterial, the required number is

$$ {}_{mn}C_n \cdot {}_{(m-1)n}C_n \cdots {}_{2n}C_n \cdot {}_nC_n \div \lfloor\underline{m}$$

$$= \frac{\lfloor\underline{mn}}{\lfloor\underline{n}\,\lfloor\underline{(m-1)\,n}} \cdot \frac{\lfloor\underline{(m-1)\,n}}{\lfloor\underline{n}\,\lfloor\underline{(m-2)\,n}} \cdots \frac{\lfloor\underline{2n}}{\lfloor\underline{n}\,\lfloor\underline{n}} \cdot \frac{\lfloor\underline{n}}{\lfloor\underline{n}} \div \lfloor\underline{m} = \frac{\lfloor\underline{mn}}{\lfloor\underline{m}\,\{\lfloor\underline{n}\}^m}.$$

21. The circle in which the nth plane cuts the sphere is cut by each of the other planes in two points, and hence the nth circle is divided into $2(n-1)$ portions; and there are two parts of the spherical surface on the two sides of each of these portions of the nth circle which would become one if the nth circle were absent. Hence the sphere is divided by n planes into $2(n-1)$ more parts than by $n-1$ planes.

Hence

$$F(n) = 2(n-1) + F(n-1). \quad \text{Similarly } F(n-1) = 2(n-2) + F(n-2), \ldots,$$
$$F(2) = 2 \cdot 1 + F(1).$$

But $F(1) = 2$, and therefore

$$F(n) = 2\{(n-1) + (n-2) + \ldots + 2 + 1\} + 2 = n(n-1) + 2.$$

22. Let the n lines pass through the point A and the m lines through the point B.

The mth line through A is cut in $n+2$ portions by the n lines through B and by the point A itself. Hence, as in 21, $F(m, n) = F(m-1, n) + n + 2$. Similarly $F(m-1, n) = F(m-2, n) + n + 2, \ldots, F(2, n) = F(1, n) + n + 2$. Also $F(1, n) = F(0, n) + n + 1$, and $F(0, n) = 2n$.

Hence $$F(m, n) = m(n+2) - 1 + 2n.$$

23. Let A, A' and B, B' be the extremities of the two diameters through which the m and n planes pass respectively. Then the circle in which the mth plane through AA' cuts the sphere is met by the n planes through BB' in $2n$ points; and therefore, counting the points A, A', the circle is divided into $2n+2$ portions. There are two parts of the sphere on the two sides of each of these portions which would become one if this mth circle were absent; the sphere would therefore be divided into $2n+2$ fewer parts if the mth circle through AA' were absent. Hence $F(m, n) = F(m-1, n) + 2n + 2$. Similarly

$$F(m-1, n) = F(m-2, n) + 2n + 2, \ldots F(2, n) = F(1, n) + 2n + 2.$$

Also $$F(1, n) = F(0, n) + 2n \text{ and } F(0, n) = 2n.$$

Hence $$F(m, n) = m \times 2n + (m-1)\,2 + 2n = 2(mn + m + n - 1).$$

24. Let the given diameters through which the a, b, c, ... planes pass respectively be AA', BB', CC', ... Any circle through AA' is cut by the $b + c + \ldots$ circles through the other diameters in $2(b + c + \ldots)$ points. Hence, as in 23,

$$F(a, b, c\ldots) = F(a-1, b, c, \ldots) + 2(b + c + \ldots) + 2;$$
$$F(a-1, b, c, \ldots) = F(a-2, b, c, \ldots) + 2(b + c + \ldots) + 2;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$F(2, b, c, \ldots) = F(1, b, c, \ldots) + 2(b + c + \ldots) + 2.$$

Also $F(1, b, c, \ldots) = F(0, b, c, \ldots) + 2(b + c + \ldots)$.

Hence $F(a, b, c, \ldots) = F(b, c, \ldots) + 2a(b + c + \ldots) + 2a - 2$.

So also $F(b, c, \ldots) = F(c, \ldots) + 2b(c + \ldots) + 2b - 2$, and so on.

Hence $F(a, b, c, \ldots) = 2\Sigma ab + 2\Sigma a - 2(n-1)$,

where $n-1$ is the number of the diameters AA', BB', &c.

25. The nth plane is cut by the others in $n-1$ straight lines. And from Art. 251, Ex. 3, these $n-1$ straight lines divide the nth plane into

$$\frac{1}{2}n(n-1)+1$$

parts; and these parts are such that if any one of them were taken away two regions into which infinite space is divided would become one.

Hence $$F(n) = F(n-1) + \frac{1}{2}n(n-1) + 1.$$

Similarly $$F(n-1) = F(n-2) + \frac{1}{2}(n-1)(n-2) + 1, \ldots,$$

$$F(2) = F(1) + \frac{1}{2}2 \cdot 1 + 1. \quad \text{Also } F(1) = 2.$$

Hence $$F(n) = \frac{1}{2}\{1 \cdot 2 + 2 \cdot 3 + \ldots (n-1)n\} + n + 1$$

$$= \frac{1}{6}(n-1)n(n+1) + n + 1 = \frac{1}{6}(n^3 + 5n + 6).$$

26. Let the points on one line be $A_1, A_2, A_3, \ldots, A_m$, and let the points on the other in the same order be $B_1, B_2, B_3, \ldots B_n$. Let $F(n, m)$ be the total number of intersections when n points on one line are joined to m points on another, the joining lines not being produced. It is easy to see from a figure that the line B_1A_r is met by $r-1$ lines through each of the points $B_2, B_3, \ldots B_n$. Hence there are $(n-1)(r-1)$ intersections on the line B_1A_r, and therefore $(n-1)\{1+2+\ldots+(m-1)\}$ altogether on lines through B_1; hence

$$F(n, m) - F(n-1, m) = (n-1)\{1+2+\ldots+(m-1)\} = \frac{1}{2}(n-1)m(m-1).$$

Similarly $$F(n-1, m) - F(n-2, m) = \frac{1}{2}(n-2)m(m-1),$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$F(2, m) - F(1, m) = \frac{1}{2} \cdot 1 \cdot m(m-1).$$

Hence, as $F(1, m) = 0$, we have

$$F(n, m) = \frac{1}{2}m(m-1)\{1+2+\ldots+(n-1)\}$$

$$= \frac{1}{2}m(m-1) \cdot \frac{1}{2}n(n-1).$$

27. The number of circles is $\frac{1}{6}n(n-1)(n-2)$. The number of pairs of circles which have none of the original n points in common is

$$\frac{1}{2}\cdot\frac{1}{6}n(n-1)(n-2)\cdot\frac{1}{6}(n-3)(n-4)(n-5).$$

Hence as such pairs of circles have two points of intersection distinct from the n given points, the whole number of such intersections is

$$\frac{1}{36}n(n-1)(n-2)(n-3)(n-4)(n-5).$$

The number of pairs of circles which have one of the original n points in common is

$$\frac{1}{2}\cdot\frac{1}{6}n(n-1)(n-2)\cdot 3\cdot\frac{1}{2}(n-3)(n-4).$$

Hence as such pairs of circles have one point of intersection distinct from the n given points, the whole number of such intersections is

$$\frac{3}{24}n(n-1)(n-2)(n-3)(n-4).$$

Hence the number required

$$=\frac{1}{72}n(n-1)(n-2)(n-3)(n-4)\{2(n-5)+9\}.$$

XXIV. Pages 303—304.

1. $(x+a)^5=x^5+5x^4a+\frac{5\cdot4}{1\cdot2}x^3a^2+\frac{5\cdot4\cdot3}{1\cdot2\cdot3}x^2a^3$

$$+\frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}xa^4+\frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5}a^5=x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5.$$

2. $(2a-x)^5=(2a)^5+5(2a)^4(-x)+\frac{5\cdot4}{1\cdot2}(2a)^3(-x)^2+\frac{5\cdot4\cdot3}{1\cdot2\cdot3}(2a)^2(-x)^3$

$$+\frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}(2a)(-x)^4+(-x)^5=32a^5-80a^4x+80a^3x^2-40a^2x^3+10ax^4-x^5.$$

3. $(1-x^2)^6=1+6(-x^2)+\frac{\lfloor 6}{\lfloor 2\,\lfloor 4}(-x^2)^2+\frac{\lfloor 6}{\lfloor 3\,\lfloor 3}(-x^2)^3+\frac{\lfloor 6}{\lfloor 4\,\lfloor 2}(-x^2)^4$

$$+\frac{\lfloor 6}{\lfloor 5\,\lfloor 1}(-x^2)^5+(-x^2)^6=1-6x^2+15x^4-20x^6+15x^8-6x^{10}+x^{12}$$

4. $(2a-3a^2)^4=(2a)^4+4\,(2a)^3(-3a^2)+\frac{4\,.\,3}{1\,.\,2}(2a)^2(-3a^2)^2$

$$+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}(2a)(-3a^2)^3+(-3a^2)^4=16a^4-96a^5+216a^6-216a^7+81a^8.$$

5. $(2x^2-3)^4=(2x^2)^4+4\,(2x^2)^3(-3)+\frac{4\,.\,3}{1\,.\,2}(2x^2)^2(-3)^2$

$$+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}(2x^2)(-3)^3+(-3)^4=16x^8-96x^6+216x^4-216x^2+81.$$

6. $(x^2-2y^3)^5=(x^2)^5+5\,(x^2)^4(-2y^3)+\frac{5\,.\,4}{1\,.\,2}(x^2)^3(-2y^3)^2$

$$+\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}(x^2)^2(-2y^3)^3+\frac{5\,.\,4\,.\,3\,.\,2}{1\,.\,2\,.\,3\,.\,4}(x^2)(-2y^3)^4+(-2y^3)^5$$

$$=x^{10}-10x^8y^3+40x^6y^6-80x^4y^9+80x^2y^{12}-32y^{15}.$$

7. The third term is

$$\frac{10\,.\,9}{1\,.\,2}x^8(-3y)^2=405x^8y^2.$$

8. The fifth term is

$$\frac{\lfloor 20}{\lfloor 4\ \lfloor 16}(3x)^{16}(-4)^4=\frac{\lfloor 20}{\lfloor 16\ \lfloor 4}3^{16}4^4x^{16}.$$

9. The twenty-first term is

$$\frac{\lfloor 22}{\lfloor 20\ \lfloor 2}2^2(-x)^{20}=924x^{20}.$$

10. The fortieth term is

$$\frac{\lfloor 42}{\lfloor 39\ \lfloor 3}x^3(-y)^{39}=-11480x^3y^{39}.$$

11. The middle term is the fifth and

$$=\frac{\lfloor 8}{\lfloor 4\ \lfloor 4}x^4=70x^4.$$

12. The middle terms are the 11th and the 12th and are respectively

$$\frac{\lfloor 21}{\lfloor 10\ \lfloor 11}x^{10}\text{ and }\frac{\lfloor 21}{\lfloor 11\ \lfloor 10}x^{11}.$$

13. The general term is

$$\frac{\lfloor n}{\lfloor r\ \lfloor n-r}x^{n-r}(-3y)^r=(-1)^r\frac{\lfloor n}{\lfloor r\ \lfloor n-r}3^rx^{n-r}y^r.$$

14. The general term is

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} (x^2)^{n-r} (y^3)^r = \frac{\lfloor n}{\lfloor r \lfloor n-r} x^{2n-2r} y^{3r}.$$

15. $(3x-2y)^{15} = (3x)^{15} + 15\,(3x)^{14}(-2y) + \frac{15 \cdot 14}{1 \cdot 2}(3x)^{13}(-2y)^2 \ldots$

$$\ldots + \frac{\lfloor 15}{\lfloor 13 \lfloor 2}(3x)^2(-2y)^{13} + \frac{\lfloor 15}{\lfloor 14 \lfloor 1}(3x)(-2y)^{14} + (-2y)^{15}.$$

16. Since n is even, the coefficient of the rth term is greatest when $r = \frac{n}{2} + 1 = 7$; and the seventh term is

$$\frac{\lfloor 12}{\lfloor 6 \lfloor 6} x^6 = 924x^6.$$

17. Since n is odd, the coefficients of the $\frac{n+1}{2}$th and the $\frac{n+3}{2}$th terms, that is of the 8th and 9th terms, are equal and are greater than any other coefficients; and these terms are respectively $6435x^7$ and $6435x^8$.

18. The coefficient of x^n in the expansion of $(1+x)^{2n}$ is $\frac{\lfloor 2n}{\lfloor n \lfloor n}$, and the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ is $\frac{\lfloor 2n-1}{\lfloor n \lfloor n-1}$.

Now $$\frac{\lfloor 2n}{\lfloor n \lfloor n} = \frac{2n \lfloor 2n-1}{n \lfloor n-1 \lfloor n} = 2\,\frac{\lfloor 2n-1}{\lfloor n-1 \lfloor n}.$$

19. The $(n+1)$th term is the middle term, and this

$$= \frac{\lfloor 2n}{\lfloor n \lfloor n} x^n = \frac{\{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)\}\{2 \cdot 4 \cdot 6 \ldots 2n\}}{\lfloor n \lfloor n} x^n$$

$$= \frac{\{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)\}\, 2^n \lfloor n}{\lfloor n \lfloor n} x^n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{\lfloor n} 2^n x^n.$$

20. $99^4 = (100-1)^4 = 100^4 - 4 \cdot 100^3 + 6 \cdot 100^2 - 4 \cdot 100 + 1 = 96059601.$

$51^4 = (50+1)^4 = 50^4 + 4 \cdot 50^3 + 6 \cdot 50^2 + 4 \cdot 50 + 1 = 6765201.$

$999^3 = (1000-1)^3 = 1000^3 - 3 \cdot 1000^2 + 3 \cdot 1000 - 1 = 997002999.$

21. $\left(x + \frac{1}{x}\right)^n = \frac{(1+x^2)^n}{x^n}$. Hence the coefficient of

x^r in $\left(x + \frac{1}{x}\right)^n$ = coefficient of x^{n+r} in $(1+x^2)^n$.

Now $x^{n+r}=(x^2)^{\frac{n+r}{2}}$, and therefore the term involving x^{n+r} is the $\left(\frac{n+r}{2}+1\right)$th, and its coefficient is $\frac{\lfloor n}{\left\lfloor n-\frac{n+r}{2}\right\rfloor \left\lfloor \frac{n+r}{2}\right.}$. There is however no term containing x^r unless $\frac{n+r}{2}$ is an integer.

22. The middle term of $\left(x-\frac{1}{x}\right)^{2n}$ is the $(n+1)$th, and this

$$=(-1)^n \frac{\lfloor 2n}{\lfloor n \lfloor n} x^n \left(\frac{1}{x}\right)^n = (-1)^n \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

23. The terms are

$$\frac{\lfloor n}{\lfloor 4 \lfloor n-4} x^4, \quad \frac{\lfloor n}{\lfloor 5 \lfloor n-5} x^5 \text{ and } \frac{\lfloor n}{\lfloor 6 \lfloor n-6} x^6$$

respectively. Hence we have

$$\frac{2 \lfloor n}{\lfloor 5 \lfloor n-5} = \frac{\lfloor n}{\lfloor 4 \lfloor n-4} + \frac{\lfloor n}{\lfloor 6 \lfloor n-6}.$$

Multiply by $\lfloor 6 \lfloor n-4$; then

$$2 \cdot 6(n-4)=5 \cdot 6+(n-5)(n-4), \text{ whence } n=7 \text{ or } n=14.$$

24. The terms are

$$nx, \quad \frac{n(n-1)}{1 \cdot 2} x^2 \text{ and } \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3$$

respectively. Hence we have

$$\frac{2n(n-1)}{\lfloor 2} = n + \frac{n(n-1)(n-2)}{\lfloor 3},$$

which gives $n=2$ or $n=7$.

The solution $n=2$ is to be rejected for there are only three terms in the expansion of $(1+x)^2$.

25. Since the second, the fourth, and all the other even terms in the expansion of $(1-x)^n$ are negative, it follows that $(1-x)^n=a-b$.

Also $(1+x)^n=a+b$. Hence $(1-x)^n(1+x)^n=(1-x^2)^n=(a-b)(a+b)$.

XXV. Pages 314—317.

1. $1-2n+3\frac{n(n-1)}{1.2}-\ldots+(-1)^n(n+1)\,1$

$=1-n+\frac{n(n-1)}{1.2}-\ldots+(-1)^n.\,1-\left\{n-2\frac{n(n-1)}{1.2}+3\frac{n(n-1)(n-2)}{1.2.3}-\ldots\right.$

$\left.+(-1)^{n-1}.\,n\right\}$

$=(1-1)^n-n(1-1)^{n-1}=0.$

2. $n-2\frac{n(n-1)}{1.2}+3\frac{n(n-1)(n-2)}{1.2.3}-\ldots\ldots+(-1)^{n-1}n$

$=n\left\{1-\frac{n-1}{1}+\frac{(n-1)(n-2)}{1.2}-\ldots+(-1)^{n-1}.1\right\}=n(1-1)^{n-1}=0.$

3. We have

$$1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\ldots\ldots+1=2^n,$$

and $1+(n-1)+\frac{(n-1)(n-2)}{1.2}+\frac{(n-1)(n-2)(n-3)}{1.2.3}+\ldots+1=2^{n-1}.$

Multiply the second by n and add to the first; then

$$1+2n+3\frac{n(n-1)}{1.2}+4\frac{n(n-1)(n-2)}{1.2.3}+\ldots=2^{n-1}(n+2).$$

4. Multiply the second formula in 3 by n and subtract the first; then

$$-1+\frac{n(n-1)}{1.2}+2\frac{n(n-1)(n-2)}{1.2.3}+\ldots+(n-1).1=n2^{n-1}-2^n;$$

$$\therefore\ c_2+2c_3+\ldots+(n-1)c_n=(n-2)2^{n-1}+1.$$

5. $c_0+3c_1+5c_2+\ldots\ldots+(2n+1)c_n$

$=c_0+c_1+c_2+\ldots\ldots+c_n+2(c_1+2c_2+3c_3+\ldots\ldots+nc_n)$

$=2^n+2n\left\{1+\frac{n-1}{1}+\frac{(n-1)(n-2)}{1.2}+\ldots+1\right\}=2^n+2n\,2^{n-1}=2^n(n+1).$

6. $3c_1+7c_2+11c_3+\ldots+(4n-1)c_n$

$=4(c_1+2c_2+3c_3+\ldots+nc_n)-(c_1+c_2+\ldots+c_n)$

$=4n\left\{1+\frac{n-1}{1}+\frac{(n-1)(n-2)}{1.2}+\ldots+1\right\}-(2^n-1)$

$=4n.\,2^{n-1}-2^n+1=2^n(2n-1)+1.$

7. $\frac{c_0}{1}+\frac{c_1}{2}+\frac{c_2}{3}+\ldots+\frac{c_n}{n+1}=\frac{1}{1}+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\frac{n(n-1)(n-2)}{1.2.3.4}+\ldots$

$$+\frac{1}{n+1}=\frac{1}{n+1}\left\{1+\frac{n+1}{1}+\frac{(n+1)n}{1.2}+\ldots+1-1\right\}=\frac{1}{n+1}(2^{n+1}-1).$$

8. $$\frac{c_0}{1}-\frac{c_1}{2}+\frac{c_2}{3}-\ldots\ldots+(-1)^n\frac{c_n}{n+1}$$

$$=\frac{1}{1}-\frac{n}{2}+\frac{n(n-1)}{1.2.3}-\ldots\ldots+\frac{(-1)^n}{n+1}$$

$$=\frac{1}{n+1}\left[-1+n+1-\frac{(n+1)n}{1.2}+\ldots\ldots+(-1)^n+1\right]$$

$$=\frac{1}{n+1}\{1-(1-1)^{n+1}\}=\frac{1}{n+1}.$$

Now add the result of 7, and we have

$$\frac{c_0}{1}+\frac{c_2}{3}+\frac{c_4}{5}+\ldots\ldots=\frac{2^n}{n+1}.$$

9. From 7 and 8 by subtraction we have

$$\frac{c_1}{2}+\frac{c_3}{4}+\ldots\ldots=\frac{2^n-1}{n+1}.$$

10. $$\frac{c_0}{2}+\frac{c_1}{3}+\frac{c_2}{4}+\ldots\ldots+\frac{c_n}{n+2}$$

$$=\frac{1}{2}+\frac{n}{1.3}+\frac{n(n-1)}{1.2.4}+\ldots\ldots+\frac{1}{n+2}$$

$$=\frac{1}{1}+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\ldots\ldots+\frac{1}{n+1}$$

$$-\left\{\frac{1}{1.2}+\frac{n}{1.2.3}+\frac{n(n-1)}{1.2.3.4}+\ldots\ldots+\frac{1}{(n+1)(n+2)}\right\}$$

$$=\frac{1}{n+1}\left\{1+\frac{n+1}{1}+\frac{(n+1)n}{1.2}+\ldots\ldots+1-1\right\}$$

$$-\frac{1}{(n+1)(n+2)}\left\{1+\frac{n+2}{1}+\frac{(n+2)(n+1)}{1.2}+\ldots+1-1-(n+2)\right\}$$

$$=\frac{1}{n+1}(2^{n+1}-1)-\frac{1}{(n+1)(n+2)}(2^{n+2}-n-3)$$

$$=\frac{n2^{n+1}+1}{(n+1)(n+2)}.$$

11. Let

$$F_n=\frac{n}{1}-\frac{1}{2}\frac{n(n-1)}{1.2}+\frac{1}{3}\frac{n(n-1)(n-2)}{1.2.3}-\ldots\ldots+(-1)^{n-1}\frac{1}{n}1;$$

then

$$F_{n+1}=\frac{n+1}{1}-\frac{1}{2}\frac{(n+1)n}{1.2}+\frac{1}{3}\frac{(n+1)n(n-1)}{1.2.3}-\ldots\ldots$$
$$+(-1)^{n-1}\frac{1}{n}.\frac{n+1}{1}+(-1)^n\frac{1}{n+1}.$$

Hence

$$F_{n+1}-F_n=\frac{1}{1}-\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}-\ldots\ldots+(-1)^{n-1}.1+(-1)^n\frac{1}{n+1}$$

$$=-\frac{1}{n+1}\left[1-\frac{n+1}{1}+\frac{(n+1)n}{1.2}-\frac{(n+1)n(n-1)}{1.2.3}+\ldots\ldots\right.$$
$$\left.+(-1)^n\frac{n+1}{1}+(-1)^{n+1}.1-1\right]$$

$$=-\frac{1}{n+1}[(1-1)^{n+1}-1]=\frac{1}{n+1}.$$

Hence $$F_{n+1}=F_n+\frac{1}{n+1}.$$

Now $F_1=1$; therefore $F_2=F_1+\frac{1}{2}=1+\frac{1}{2}$; then $F_3=F_2+\frac{1}{3}=1+\frac{1}{2}+\frac{1}{3}$; and so on, so that $F_n=\frac{1}{1}+\frac{1}{2}+\ldots\ldots+\frac{1}{n}$.

12. We have, from Art. 259, Ex. 3,

$$\frac{c_0}{x}-\frac{c_1}{x+1}+\frac{c_2}{x+2}-\ldots\ldots+(-1)^n\frac{c_n}{x+n}=\frac{\lfloor n}{x(x+1)\ldots(x+n)}.$$

Put $x=\frac{1}{3}$; and we have

$$\frac{3c_0}{1}-\frac{3c_1}{4}+\frac{3c_2}{7}-\ldots\ldots+(-1)^n\frac{3c_n}{3n+1}=\frac{3^{n+1}\lfloor n}{1.4.7\ldots(3n+1)};$$

$$\therefore\ \frac{c_0}{1}-\frac{c_1}{4}+\frac{c_2}{7}-\ldots\ldots+(-1)^n\frac{c_n}{3n+1}=\frac{3^n\lfloor n}{1.4.7\ldots(3n+1)}.$$

13. We have

$$(1+x)^n=c_0+c_1x+c_2x^2+\ldots\ldots+c_{n-r}x^{n-r}+\ldots\ldots+c_nx^n.$$

Also, from Art. 256,

$$(1+x)^n=c_n+c_{n-1}x+c_{n-2}x^2+\ldots\ldots+c_rx^{n-r}+\ldots\ldots+c_0x^n.$$

The coefficient of x^{n-r} in the product of the two series

$$= c_0c_r + c_1c_{r+1} + c_2c_{r+2} + \ldots\ldots + c_{n-r}c_n.$$

Also the coefficient of x^{n-r} in $(1+x)^n \times (1+x)^n$, that is in $(1+x)^{2n}$, is $\frac{\lfloor 2n}{\lfloor n-r \lfloor n+r}$. But these coefficients are equal by Art. 91.

14. The coefficient of x^{r-1} in

$$n(1+x)^{n-1} \text{ is } n\,\frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r} = r \cdot \frac{\lfloor n}{\lfloor r \lfloor n-r} = r \cdot c_r.$$

Hence $$n(1+x)^{n-1} = c_1 + 2c_2x + 3c_3x^2 + \ldots\ldots + nc_nx^{n-1} \ldots\ldots\ldots\ldots\text{(i)}.$$

Multiply by x and add to the expansion of $(1+x)^n$; then

$$(1+x)^{n-1}\{1+(n+1)x\} = c_0 + 2c_1x + 3c_2x^2 + \ldots\ldots + (n+1)c_nx^n \ldots\text{(ii)}.$$

We have also

$$(1+x)^n = c_n + c_{n-1}x + \ldots\ldots + c_{n-r}x^r + \ldots\ldots + c_0x^n \ldots\ldots\ldots\text{(iii)}.$$

The coefficient of x^{n-1} in the product of the series in (i) and (iii) is

$$c_1^2 + 2c_2^2 + \ldots\ldots + nc_n^2,$$

and the coefficient of x^{n-1} in $n(1+x)^{2n-1}$ is $n\,\frac{\lfloor 2n-1}{\lfloor n-1 \lfloor n}$.

Hence $$\Sigma\, rc_r^2 = \frac{\lfloor 2n-1}{\lfloor n-1 \lfloor n-1}.$$

Also the coefficient of x^n in the product of the series in (ii) and (iii) is $c_0^2 + 2c_1^2 + \ldots + (n+1)c_n^2$, and the coefficient of x^n in $(1+x)^{2n-1}\{1+(n+1)x\}$ is

$$\frac{\lfloor 2n-1}{\lfloor n \lfloor n-1} + (n+1)\,\frac{\lfloor 2n-1}{\lfloor n-1 \lfloor n}.$$

Hence $$\Sigma\,(r+1)c_r^2 = \frac{(n+2)\lfloor 2n-1}{\lfloor n \lfloor n-1}.$$

15. We have

$$\{(1+x)^n - 1\}^m = (1+x)^{mn} - {}_mC_1(1+x)^{(m-1)n} + {}_mC_2(1+x)^{(m-2)n} - \ldots + (-1)^{m-1}{}_mC_{m-1}(1+x)^n + (-1)^m.$$

The coefficient of x^m on the right

$$= (-1)^{m-1}\{{}_{mn}C_1 \cdot {}_nC_m - {}_mC_2 \cdot {}_{2n}C_m + {}_mC_3 \cdot {}_{3n}C_m - \ldots + (-1)^{m-1}{}_mC_m \cdot {}_{mn}C_m\}.$$

But $$\{(1+x)^n - 1\}^m = \{nx + \frac{n(n-1)}{1\,.\,2}x^2 + \ldots\}^m,$$

and the coefficient of x^m in this is n^m.

Hence $$n^m = (-1)^{m-1}\{{}_mC_1 \cdot {}_nC_m - {}_mC_2 \cdot {}_{2n}C_m + \ldots\}.$$

16. We have

$$1-n+\frac{n(n-1)}{1\cdot 2}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\ldots+(-1)^n=0 \quad \ldots\ldots\ldots \text{(i)}.$$

Change n into $n-1$; then if $n>1$

$$1-(n-1)+\frac{(n-1)(n-2)}{1\cdot 2}-\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}+\ldots+(-1)^{n-1}=0 \ldots \text{(ii)}.$$

Multiply (i) by a and (ii) by n and add; then

$$a-n(a-1)+\frac{n(n-1)}{1\cdot 2}(a-2)-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(a-3)+\ldots$$
$$+(-1)^n(a-n)=0 \ldots \text{(iii)}.$$

Now change a into $a-1$ and n into $n-1$; then, n being greater than 2, we have

$$a-1-(n-1)(a-2)+\frac{(n-1)(n-2)}{1\cdot 2}(a-3)-\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}(a-4)$$
$$+\ldots+(-1)^{n-1}(a-n)=0 \ldots \text{(iv)}.$$

Multiply (iii) by b and (iv) by n and add; then

$$ab-n(a-1)(b-1)+\frac{n(n-1)}{1\cdot 2}(a-2)(b-2)-\ldots$$
$$+(-1)^n(a-n)(b-n)=0 \ldots\ldots \text{(v)}.$$

Now change a into $a-1$, b into $b-1$ and n into $n-1$; then, n being greater than 3, we have

$$(a-1)(b-1)-(n-1)(a-2)(b-2)+\frac{(n-1)(n-2)}{1\cdot 2}(a-3)(b-3)-\ldots$$
$$+(-1)^{n-1}(a-n)(b-n)=0 \ldots\ldots \text{(vi)}.$$

Multiply (v) by c and (vi) by n and add; then

$$abc-n(a-1)(b-1)(c-1)+\frac{n(n-1)}{1\cdot 2}(a-2)(b-2)(c-2)-\ldots$$
$$+(-1)^n(a-n)(b-n)(c-n)=0.$$

17. Since the coefficients of the terms equidistant respectively from the beginning and the end are equal, every coefficient except the middle one, when there is one, occurs twice; and therefore the sum of all the coefficients, exclusive of the middle one, is even. But the sum of all the coefficients is even [Art. 259, I], whence it follows that the coefficient of the middle term, whenever there is one, must be even.

18. $\{x^2+(a+b)x+ab\}^n=(x+a)^n\times(x+b)^n$

$$=\{x^n+{}_nC_1x^{n-1}a+{}_nC_2x^{n-2}a^2+\ldots+{}_nC_rx^{n-r}a^r+\ldots+a^n\}$$
$$\times\{b^n+{}_nC_1b^{n-1}x+{}_nC_2b^{n-2}x^2+\ldots+{}_nC_rb^{n-r}x^r+\ldots+x^n\},$$

and the coefficient of x^n in the product of the two series is

$$a^n+{}_nC_1{}^2a^{n-1}b+\ldots+{}_nC_r{}^2a^rb^{n-r}+\ldots+{}_nC_1{}^2ab^{n-1}+b^n.$$

19. $P_n = \frac{\lfloor n}{\lfloor 1 \lfloor n-1} \cdot \frac{\lfloor n}{\lfloor 2 \lfloor n-2} \cdots \frac{\lfloor n}{\lfloor n-1 \lfloor 1} = \frac{(\lfloor n)^{n-1}}{\{\lfloor 2 \lfloor 3 \ldots \lfloor n-1\}^2}$.

Also $P_{n+1} = \frac{(\lfloor n+1)^n}{\{\lfloor 2 \lfloor 3 \ldots \lfloor n\}^2}$. Hence $\frac{P_{n+1}}{P_n} = \frac{\lfloor n\,(n+1)^n}{(\lfloor n)^2} = \frac{(n+1)^n}{\lfloor n}$.

20. $(1-x)^n = (1+x-2x)^n = (1+x)^n - n \cdot 2x\,(1+x)^{n-1}$

$$+\frac{n\,(n-1)}{1\,.\,2}\,2^2x^2\,(1+x)^{n-2} - \ldots$$

$$= (1+x)^n - 2nx\,(1+x)^{n-1} + \frac{2n\,(2n-2)}{1\,.\,2}\,x^2\,(1+x)^{n-2} - \ldots$$

21. The series may be written

$$1 - n\,\frac{1}{1+nx} + \frac{n\,(n-1)}{1\,.\,2}\,\frac{1}{(1+nx)^2} - \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,\frac{1}{(1+nx)^3} + \ldots$$

$$-\frac{nx}{1+nx}\left\{1 - \frac{n-1}{1}\,\frac{1}{1+nx} + \frac{(n-1)\,(n-2)}{1\,.\,2}\,\frac{1}{(1+nx)^2} - \ldots\right\}$$

$$= \left(1 - \frac{1}{1+nx}\right)^n - \frac{nx}{1+nx}\left(1 - \frac{1}{1+nx}\right)^{n-1} = 0.$$

22. The general term of the expansion of

$$(a+b+c+d+e)^5 \text{ is } \frac{\lfloor 5}{\lfloor p \lfloor q \lfloor r \lfloor s \lfloor t}\,a^p b^q c^r d^s e^t,$$

where each of p, q, r, s, t is zero or a positive integer and $p+q+r+s+t=5$.

Hence the coefficients of all terms of the type a^5 is 1, of the type a^4b is $\frac{\lfloor 5}{\lfloor 4 \lfloor 1} = 5$, of the type a^3b^2 is $\frac{\lfloor 5}{\lfloor 3 \lfloor 2} = 10$, of the type a^3bc is $\frac{\lfloor 5}{\lfloor 3 \lfloor 1 \lfloor 1} = 20$, of the type a^2b^2c is $\frac{\lfloor 5}{\lfloor 2 \lfloor 2 \lfloor 1} = 30$, of the type a^2bcd is $\frac{\lfloor 5}{\lfloor 2 \lfloor 1 \lfloor 1 \lfloor 1} = 60$, and the coefficient of $abcde$ is $\frac{\lfloor 5}{\lfloor 1 \lfloor 1 \lfloor 1 \lfloor 1 \lfloor 1} = 120$.

23. We have

$$(1-x)^n = 1 - nx + \frac{n\,(n-1)}{1\,.\,2}x^2 - \ldots + (-1)^r\,\frac{\lfloor n}{\lfloor r \lfloor n-r}\,x^r + \ldots,$$

and $$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots$$

The coefficient of x^r in the product of the two series is

$$a_r - na_{r-1} + \frac{n\,(n-1)}{1\,,\,2}\,a_{r-2} - \ldots + (-1)^r\,\frac{\lfloor n}{\lfloor r \lfloor n-r}\,a_0.$$

Hence the above series is equal to the coefficient of x^r in

$$(1+x+x^2)^n \times (1-x)^n,$$

that is in $(1-x^3)^n$. But there is clearly no term in the expansion of $(1-x^3)^n$ in which the index of x is not a multiple of 3; and therefore $a_r - na_{r-1} + \ldots = 0$, unless r is a multiple of 3.

24. Let $(1+x+x^2+\ldots x^r)^n = a_0 + a_1x + a_2x^2 + \ldots$

$$+ a_{nr-2}x^{nr-2} + a_{nr-1}x^{nr-1} + a_{nr}x^{nr} \ldots\ldots \text{(i)}.$$

Then $(1+x^{-1}+x^{-2}+\ldots+x^{-r})^n$

$$= a_0 + a_1x^{-1} + a_2x^{-2} + \ldots + a_{nr-2}x^{2-nr} + a_{nr-1}x^{1-nr} + a_{nr}x^{-nr},$$

or, multiplying by x^{nr},

$$(1+x+x^2+\ldots+x^r)^n = a_0x^{nr} + a_1x^{nr-1} + a_2x^{nr-2} + \ldots$$

$$+ a_{nr-2}x^2 + a_{nr-1}x + a_{nr} \ldots \text{(ii)}.$$

Now, from Art. 91, the coefficients of the same powers of x in the two expansions (i) and (ii) must be equal. Hence

$$a_0 = a_{nr},\ a_1 = a_{nr-1},\ a_2 = a_{nr-2},\ \&c.$$

25. Let $(1+x+x^2)^n = a_0 + a_1x + \ldots + a_rx^r + \ldots + a_{2n}x^{2n}$,

then $(1-x+x^2)^n = a_0 - a_1x + \ldots + (-1)^r a_r x^r + \ldots + a_{2n}x^{2n}$,

and, from 24, $(1+x+x^2)^n = a_{2n} + a_{2n-1}x + \ldots + a_1x^{2n-1} + a_0x^{2n}$.

Hence $a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2$ = coefficient of x^{2n} in $(1-x+x^2)^n(1+x+x^2)^n$, that is in $(1+x^2+x^4)^n$. And it is obvious that the coefficient of x^{2n} in $(1+x^2+x^4)^n$ is the same as the coefficient of x^n in $(1+x+x^2)^n = a_n$.

Thus $a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2 = a_n$.

Since $a_0 = a_{2n},\ a_1 = a_{2n-1}$, &c.

$$a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2 = 2\{a_0^2 - a_1^2 + \ldots + (-1)^{n-1}a_{n-1}^2\} + (-1)^n a_n^2 = a_n.$$

26. Let $(1+x+x^2)^n = a_0 + a_1x + \ldots\ldots + a_rx^r + \ldots\ldots + a_{2n}x^{2n}$.

Then, from 24,

$$(1-x+x^2)^n = a_{2n} - a_{2n-1}x + \ldots\ldots + (-1)^r a_{2n-r}x^r + \ldots\ldots + a_0x^{2n}.$$

Hence $-a_0a_1 + a_1a_2 - a_2a_3 - \ldots\ldots - a_{2n-1}a_{2n}$

is the coefficient of x^{2n-1} in the expansion of $(1+x+x^2)^n \times (1-x+x^2)^n$, that is of $(1+x^2+x^4)^n$; and it is obvious that only *even* powers of x occur in the expansion of $(1+x^2+x^4)^n$. Hence $a_0a_1 - a_1a_2 + \ldots\ldots = 0$.

27. The general term of the expansion is

$$\frac{\lfloor n}{\lfloor\alpha\ \lfloor\beta\ \lfloor\gamma\ldots}\ a_1^{\alpha}a_2^{\beta}a_3^{\gamma}\ldots\ldots,$$

where each of the letters α, β, γ,..... is zero or a positive integer and $\alpha+\beta+\gamma+\ldots\ldots=n$. Hence, if all the letters are zero or unity, the coefficient of the term will be $\lfloor n$.

28. Let $$(1+x)(1+x+x^2)\ldots(1+x+x^2+\ldots x^n)$$
$$=a_0+a_1x+\ldots+a_{\frac{1}{2}n(n+1)}x^{\frac{1}{2}n(n+1)}.$$

Then $$(1+x^{-1})(1+x^{-1}+x^{-2})\ldots(1+\ldots+x^{-n})$$
$$=a_0+a_1x^{-1}+\ldots+a_{\frac{1}{2}n(n+1)}x^{-\frac{1}{2}n(n+1)}.$$

Multiply by $x^{\frac{1}{2}n(n+1)}$; then we have
$$(1+x)\ldots(1+x+\ldots+x^n)=a_{\frac{1}{2}n(n+1)}+\ldots+a_0x^{\frac{1}{2}n(n+1)}.$$

Hence $$a_0=a_{\frac{1}{2}n(n+1)},\ a_1=a_{\frac{1}{2}n(n+1)-1},\ \&c.$$

Put $x=1$; then
$$2\ .\ 3\ldots\ldots(n+1)=O+E,$$
where O is the sum of all the odd coefficients, and E the sum of the even coefficients. Again, put $x=-1$; then $0=O-E$.

Hence $$O=E=\frac{1}{2}\lfloor n+1.$$

29. $$(1+x+x^2)^n=\{1+x(1+x)\}^n$$
$$=x^n(1+x)^n+n\ .\ x^{n-1}(1+x)^{n-1}+\frac{n(n-1)}{1\ .\ 2}x^{n-2}(1+x)^{n-2}+\ldots$$

And the coefficient of x^n in the series
$$=1+\frac{n(n-1)}{1^2}+\frac{n(n-1)}{1\ .\ 2}\ \frac{(n-2)(n-3)}{1\ .\ 2}-\&c.$$

30. The number of ways will be the coefficient of x^{18} in the expansion of $(x^1+x^3+x^5+x^7+\ldots)^8$, and this is the same as the coefficient of x^{10} in the expansion of $(1+x^2+x^4+\ldots+x^{10})^8$. This coefficient can be found by the method of Art. 263. but the result is much more easily obtained by assuming the Binomial Theorem to be true for a negative exponent; we then have

$$\left(\frac{1-x^{12}}{1-x^2}\right)^8=(1-8x^{12}+\text{higher powers})$$
$$\left\{1+8x^2+\ldots+\frac{8\ .\ 9\ .\ 10\ .\ 11\ .\ 12}{\lfloor 5}x^{10}+\text{higher powers}\right\}$$
$$=1+\ldots+\frac{8\ .\ 9\ .\ 10\ .\ 11\ .\ 12}{\lfloor 5}x^{10}+\ldots,$$

so that the required coefficient is 792.

XXVI. Pages 332—333.

1. The series is $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, and is therefore convergent by Theorem V.

2. The series is $\frac{1}{b}\left\{\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\ldots\right\}$, and is therefore convergent by Theorem V.

3. $\frac{u_{n+1}}{u_n}=\frac{n+3}{2n+4}<\frac{2}{3}$. Hence the series is convergent. [Theorem III.]

4. $\frac{u_{n+1}}{u_n}=\frac{2n+3}{3n+4}<\frac{3}{4}$. Hence the series is convergent. [Theorem III.]

5. $\frac{u_{n+1}}{u_n}=\frac{2n+1}{3n+3}<\frac{2}{3}$. Hence the theorem is convergent. [Theorem III.]

6. Comparing the given series with the known divergent series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$, we see that the ratio of the nth terms is $\frac{n}{n+x-1}$, and this ratio is finite for all values of n. The given series is therefore divergent. [Theorem II.]

7. Comparing the given series with the known divergent series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$, we see that the ratio of the nth terms is

$$\frac{n}{1+nx}=\frac{1}{x}-\frac{1}{x(1+nx)},$$

and this ratio is finite for all values of n. The given series is therefore divergent. [Theorem II.]

8. The series is less than $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots$, and is therefore convergent if $x>1$.

If $x=1$, the series becomes $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$, and is obviously divergent.

If $x<1$, the series is greater than the series $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$, and is therefore divergent.

9. If $x=1$, the series becomes $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$, and is therefore divergent.

The series is less than $1+x+x^2+x^3+\ldots$, which is convergent if x be less than 1; the given series is therefore itself convergent if $x<1$.

The given series is also less than the series $\frac{1}{x}+\frac{x}{x^2}+\frac{x^2}{x^4}+\ldots+\frac{x^n}{x^{2n}}+\ldots$, which is convergent if $x>1$; the given series is therefore itself convergent if $x>1$.

10. If $x=1$, the series becomes $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$, which is known to be divergent.

If $x<1$, $\frac{1}{1+nx^n}>\frac{1}{1+n}$, and hence the given series is greater than the known divergent series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$

Also $\frac{1}{1+nx^n}<\frac{1}{x^n}$. Hence the series is less than the series

$$\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^n}+\ldots,$$

which is known to be convergent when $x>1$. Hence the given series is convergent when $x>1$.

11. Here $\frac{u_{n+1}}{u_n}=\frac{(n+1)(n+2)}{(n+2)(n+3)}x.$

Hence $$u_{n+1}:u_n \gtrless 1 \text{ as } x \gtrless \frac{n+3}{n+1} \gtrless 1+\frac{2}{n+1}.$$

Now if $x>1$ it will be $>1+\frac{2}{n+1}$ either from the beginning or after a finite number of terms, and if $x<1$, then $x<1+\frac{2}{n+1}$.

Hence the series is convergent or divergent according as x is less or greater than 1.

If $x=1$, the series becomes $\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}+\ldots$, which is equal to unity.

12. If $x\neq 1$ the terms are alternately positive and negative, each term is less than the preceding, and the terms diminish without limit. The series is therefore convergent. [Theorem V.]

If $x>1$, the separate terms increase indefinitely with n, and the series is divergent.

13. $u_n>\frac{1}{n}$. Hence $S>\frac{1}{1}+\frac{1}{2}+\ldots\frac{1}{n}+\ldots$

14. $u_n=\frac{2n-1}{2n^2-2n+1}>\frac{1}{n}$. Hence $S>\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}+\ldots$

15. $\frac{u_{n+1}}{u_n}=\frac{x+mn}{x+m(n+1)}m=\left\{1-\frac{m}{x+(n+1)m}\right\}m.$

Hence, either from the beginning or after a finite number of terms, $u_{n+1}:u_n \gtrless 1$ according as $m \gtrless 1$. The series is therefore convergent if $m<1$ and divergent if $m>1$.

If $m=1$, the series becomes $\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\ldots$, which is the series considered in question 6.

16. The series is less than $\frac{1}{x}+\frac{m}{x}+\frac{m^2}{x}+\ldots$, and is therefore convergent when $m<1$.

If $m=1$, the series becomes $\frac{1}{x+1}+\frac{1}{x+1}+\frac{1}{x+1}+\ldots$, and is obviously divergent.

If $m>1$, m^r will be greater than x for some finite value of r, and for this and all greater values of r each term of the given series will be greater than $\frac{1}{2}$, and hence the whole series is divergent when $m>1$.

17. It is obvious that $u_n > \frac{n^2}{n^3+3n^2+2n} > \frac{1}{6n}$.

Hence $$6S > \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

18. $$u_n > \frac{n}{2n\sqrt{(n+1)}} > \frac{1}{2\sqrt{(n+1)}}.$$

Hence $$2S > \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{(n+1)}}+\ldots,$$

and the latter series is known to be divergent.

19. $u_n > \frac{1}{2\sqrt{n}}$. Hence $2S > \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots$

20. $u_n = \frac{\sqrt{n}-1}{(n+1)\sqrt{n}} > \frac{1}{2(n+1)}$, if $n>4$. Hence $2S > \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$

21. $u_n = \frac{1}{\sqrt{(n^2+1)}+n} > \frac{1}{3n}$. Hence $3S_n > \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots$

22. $$\frac{u_{n+1}}{u_n} = \left(\frac{2n+1}{2n+3}\right)^k x = \left(1-\frac{2}{2n+3}\right)^k x.$$

Hence, either from the beginning or after a finite number of terms, $u_{n+1}:u_n$ is greater or less than 1 according as x is greater or less than 1. The series is therefore convergent or divergent according as x is less or greater than unity.

If $x=1$, the series becomes $\frac{1}{1^k}+\frac{1}{3^k}+\frac{1}{5^k}+\ldots < \frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\ldots,$

and the latter series is known to be convergent if $k>1$.

If $k=1$, the series becomes $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ldots > \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right)$ and is therefore divergent.

If $k<1$, each term of the series $\Sigma \frac{1}{(2r+1)^k}$ is greater than the corresponding term of $\Sigma \frac{1}{2r+1}$, and therefore $\Sigma \frac{1}{(2r+1)^k}$ is divergent if $k<1$.

23. The series is less than $1+x+x^2+\ldots+x^n+\ldots$, and is therefore convergent when $x<1$.

If $x=1$, the series becomes $\Sigma \frac{2n}{n^2+1}$ which is greater than $\Sigma \frac{1}{n}$. Hence the series is divergent if $x=1$.

If $x>1$, $\Sigma \frac{2nx^n}{n^2+1} > \Sigma \frac{2n}{n^2+1}$, and is therefore divergent.

24. $$\frac{u_{n+1}}{u_n} = \frac{2n-3}{2n-5} \frac{n^3-5n}{(n+1)^3-5(n+1)} x = \frac{2n^4-3n^3-10n^2+15n}{2n^4+n^3-19n^2+2n+20} x.$$

Hence the limit of $u_{n+1} : u_n$ is x. Hence, from and after some finite value of n, $u_{n+1} : u_n$ will be greater or less than 1 according as x is greater or less than 1. The series is therefore convergent when $x<1$ and divergent when $x>1$.

If $x=1$, the series becomes $\Sigma (2n-5)/(n^3-5n)$ which is less than $\Sigma 4/n^2$, and $\Sigma 1/n^2$ is convergent.

25. Let $r^2 > 2x$, then $\Sigma_r^\infty 1/(n^2-x) < \Sigma_r^\infty 2/n^2$. Also $\Sigma_1^r 1/(n^2-x)$ is finite, since r is finite and x not the square of an integer.

Hence, as $\Sigma 1/n^2$ is convergent, the whole series is convergent.

26. $$u_n = \left(\frac{1}{\sqrt{(n-1)}} - \frac{1}{\sqrt{n}}\right) n^m x^n = n^m x^n \left(\frac{1}{n-1} - \frac{1}{n}\right) \Big/ \left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}}\right)$$

$$= \tfrac{1}{2} n^{m-\frac{3}{2}} x^n \text{ ultimately.}$$

Hence $u_{n+1}/u_n = x$ ultimately. Therefore Σu_n is convergent if $x<1$, and divergent if $x>1$. If $x=1$, by Art. 274 Σu_n is convergent or divergent according as m is or is not less than $\frac{1}{2}$.

27. $$u_n = n^{k+\frac{1}{2}} \left\{ \sqrt{\left(1-\frac{1}{n}\right)} - 2\sqrt{\left(1-\frac{2}{n}\right)} + \sqrt{\left(1-\frac{3}{n}\right)} \right\} x^n$$

$$= n^{k+\frac{1}{2}} \left(-\frac{1}{4} n^2\right) x^n \text{ ultimately } = -\frac{1}{4} n^{k-\frac{3}{2}} x^n.$$

Hence the conditions for convergency and divergency are the same as in 26.

XXVII. Pages 341—342.

1. (i) $\dfrac{(-2)(-3)\ldots(-r-1)}{\lfloor r}(-x)^r=(r+1)\,x^r.$

(ii) $\dfrac{(-3)(-4)\ldots(-r-2)}{\lfloor r}(-x)^r=\dfrac{(r+1)(r+2)}{1\cdot 2}\,x^r.$

(iii) $\dfrac{(-n)(-n-1)\ldots(-n-r+1)}{\lfloor r}(-x)^r=\dfrac{n(n+1)\ldots(n+r-1)}{\lfloor r}\,x^r.$

(iv) $\dfrac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\ldots\left(-\frac{2}{3}-r+1\right)}{\lfloor r}x^r=(-1)^r\,\dfrac{2\cdot 5\ldots(3r-1)}{3^r\,\lfloor r}\,x^r.$

(v) $\dfrac{\frac{2}{3}\left(\frac{2}{3}-1\right)\ldots\left(\frac{2}{3}-r+1\right)}{\lfloor r}x^r=(-1)^{r-1}\,\dfrac{2\cdot 1\cdot 4\cdot 7\ldots(3r-5)}{3^r\,\lfloor r}\,x^r.$

(vi) $\dfrac{\frac{5}{3}\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)\ldots\left(\frac{5}{3}-r+1\right)}{\lfloor r}x^r=(-1)^{r-2}\,\dfrac{5\cdot 2\cdot 1\cdot 4\ldots(3r-8)}{3^r\,\lfloor r}\,x^r.$

(vii) $\dfrac{\left(-\frac{3}{5}\right)\left(-\frac{3}{5}-1\right)\ldots\left(-\frac{3}{5}-r+1\right)}{\lfloor r}(-5x)^r=\dfrac{3\cdot 8\cdot 13\ldots(5r-2)}{\lfloor r}\,x^r.$

(viii) $\dfrac{\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)\ldots\left(\frac{2}{5}-r+1\right)}{\lfloor r}(-5x)^r=-\dfrac{2\cdot 3\cdot 8\ldots(5r-7)}{\lfloor r}\,x^r.$

(ix) $\dfrac{\left(-\frac{q}{p}\right)\left(-\frac{q}{p}-1\right)\ldots\left(-\frac{q}{p}-r+1\right)}{\lfloor r}(-x)^r$

$$=\frac{q\,(q+p)\,(q+2p)\ldots(q+\overline{r-1}p)}{p^r\,\lfloor r}\,x^r.$$

(x) $(2a+3x)^{-\frac{2}{5}}=(2a)^{-\frac{2}{5}}\left(1+\dfrac{3x}{2a}\right)^{-\frac{2}{5}}$

$$=\ldots+(2a)^{-\frac{2}{5}}\,\frac{\left(-\frac{2}{5}\right)\left(-\frac{2}{5}-1\right)\ldots\left(-\frac{2}{5}-r+1\right)}{\lfloor r}\left(\frac{3x}{2a}\right)^r$$

$$=\ldots(-1)^r\,\frac{2\cdot 7\cdot 12\ldots(5r-3)}{5^r\,\lfloor r}\,\frac{3^r x^r}{(2a)^{r+\frac{2}{5}}}.$$

(xi) $(a^2-2ax)^{\frac{5}{2}}=(a^2)^{\frac{5}{2}}\left(1-\frac{2x}{a}\right)^{\frac{5}{2}}$

$$=a^5\left\{1+\ldots+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\ldots\left(\frac{5}{2}-r+1\right)}{\lfloor r}\left(-\frac{2x}{a}\right)^r+\ldots\right\}$$

$$=a^5\left\{1+\ldots-\frac{5\,.\,3\,.\,1\,.\,1\,.\,3\,.\,5\ldots(2r-7)}{\lfloor r}\left(\frac{x}{a}\right)^r+\ldots\right\}.$$

(xii) $(4-7x)^{\frac{2}{7}}=4^{\frac{2}{7}}\left(1-\frac{7x}{4}\right)^{\frac{2}{7}}$

$$=4^{\frac{2}{7}}\left\{1+\ldots+\frac{\frac{2}{7}\left(\frac{2}{7}-1\right)\ldots\left(\frac{2}{7}-r+1\right)}{\lfloor r}\left(-\frac{7x}{4}\right)^r+\ldots\right\}$$

$$=\ldots-\frac{2\,.\,5\,.\,12\ldots(7r-9)}{\lfloor r}4^{-r+\frac{2}{7}}x^r\ldots$$

2. (i) $\dfrac{\frac{27}{4}\cdot\frac{23}{4}\cdot\frac{19}{4}\cdot\frac{15}{4}\cdot\frac{11}{4}\cdot\frac{7}{4}\cdot\frac{3}{4}\cdot-\frac{1}{4}}{\lfloor 8}\left(\frac{4}{3}x\right)^8$

and (ii) $\dfrac{\frac{21}{4}\cdot\frac{17}{4}\cdot\frac{13}{4}\cdot\frac{9}{4}\cdot\frac{5}{4}\cdot\frac{1}{4}\cdot-\frac{3}{4}}{\lfloor 7}\left(\frac{2}{3}x\right)^7$.

3. The rth term is $\begin{matrix}>\\=\\<\end{matrix}$ $(r+1)$th term according as

$$r\begin{matrix}>\\=\\<\end{matrix}\frac{-(1+n)x}{1-x}\begin{matrix}>\\=\\<\end{matrix}\frac{11\cdot\frac{7}{9}}{1-\frac{7}{9}}\begin{matrix}>\\=\\<\end{matrix}38\tfrac{1}{2}.$$

Hence the 39th term is the greatest.

4. The rth term is $\begin{matrix}>\\=\\<\end{matrix}$ $(r+1)$th term according as

$$r\begin{matrix}>\\=\\<\end{matrix}\frac{-(1+n)\cdot\frac{2}{3}x}{1-\frac{2}{3}x}\begin{matrix}>\\=\\<\end{matrix}\frac{\frac{1}{2}}{1-\frac{1}{2}}\begin{matrix}>\\=\\<\end{matrix}1.$$

Hence the first and second terms are equal to one another and are greater than any other terms.

5. The rth term will be greater than the succeeding provided

$$r > \frac{(n+1)x}{x+1} > \frac{\frac{127}{5} \cdot \frac{5}{6}}{\frac{5}{6}+1} > 11\frac{6}{11}.$$

Hence each term after the 11th is greater than the succeeding term.

6. $(1-x)^{-3} = 1 + 3x + \frac{3 \cdot 4}{1 \cdot 2}x^2 + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3}x^3 + \ldots$

$$= \frac{1}{2}\{1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \ldots + (r+1)(r+2)x^r + \ldots\}.$$

Hence

$$\frac{19-21x}{(1-x)^3} = \frac{1}{2}(19-21x)\{1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \ldots + (r+1)(r+2)x^r + \ldots\}.$$

The coefficient of x^r is

$$\frac{1}{2}\{19(r+1)(r+2) - 21r(r+1)\} = 19 + 18r - r^2 = 100 - (r-9)^2,$$

which is positive so long as r is less than 19, and is greatest when $r=9$.

7. If a_1 be the $(r+1)$th coefficient in the expansion of $(1+x)^n$, we have

$$a_2 = a_1 \cdot \frac{n-r}{r+1},\ a_3 = a_2 \cdot \frac{n-r-1}{r+2} \text{ and } a_4 = a_3 \cdot \frac{n-r-2}{r+3}.$$

Hence $\frac{a_1}{a_1+a_2} = \frac{r+1}{n+1}$, $\frac{a_2}{a_2+a_3} = \frac{r+2}{n+1}$ and $\frac{a_3}{a_3+a_4} = \frac{r+3}{n+1}$.

8. (i) $\frac{a}{\sqrt[3]{(a^2-x^2)}} = a \cdot a^{-\frac{2}{3}}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{3}}$

$$= a^{\frac{1}{3}}\left\{1 + \ldots + \frac{1 \cdot 4 \cdot 7 \ldots (3r-2)}{3 \cdot 6 \cdot 9 \ldots 3r}\left(\frac{x}{a}\right)^{2r} + \ldots\right\}$$

(ii) $\frac{a+x}{a-x} = \left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right)^{-1} = \left(1+\frac{x}{a}\right)\left(1+\frac{x}{a}+\ldots+\frac{x^r}{a^r}+\ldots\right)$

$$= 1 + \frac{2x}{a} + \ldots + \frac{2x^r}{a^r} + \ldots$$

(iii) $\left(\frac{a+x}{a-x}\right)^2 = \left(1+\frac{x}{a}\right)^2\left(1-\frac{x}{a}\right)^{-2}$

$$= \left(1+\frac{2x}{a}+\frac{x^2}{a^2}\right)\left\{1+\frac{2x}{a}+\frac{3x^2}{a^2}+\ldots+(r+1)\frac{x^r}{a^r}+\ldots\right\}$$

$$= 1 + \frac{4x}{a} + \frac{8x^2}{a^2} + \ldots + \frac{x^r}{a^r}\{r+1+2r+r-1\} + \ldots$$

(iv) $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} = \left(1+\frac{x}{a}\right)\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$

$$= \left(1+\frac{x}{a}\right)\left\{1+\frac{1}{2}\frac{x^2}{a^2}+\frac{1\cdot 3}{2\cdot 4}\frac{x^4}{a^4}+\ldots+\frac{1\cdot 3\cdot 5\ldots(2r-1)}{2\cdot 4\cdot 6\ldots 2r}\frac{x^{2r}}{a^{2r}}+\ldots\right\}.$$

(v) $(a+x)^2(a-x)^{-3} = \frac{1}{a}\left(1+2\frac{x}{a}+\frac{x^2}{a^2}\right)\left\{1+3\frac{x}{a}+\frac{3\cdot 4}{1\cdot 2}\frac{x^2}{a^2}+\ldots\right.$

$$\left.\ldots+\frac{(r+1)(r+2)}{1\cdot 2}\frac{x^r}{a^r}+\ldots\right\}.$$

Hence the coefficient of x^r

$$= \frac{1}{2a^{r+1}}\{(r+1)(r+2)+2r(r+1)+(r-1)r\} = (2r^2+2r+1)\,a^{-r-1}.$$

(vi) $(a-x)^4(a+x)^{-3} = a^2\left(1-\frac{x}{a}\right)^4\left(1+\frac{x}{a}\right)^{-2}$

$$= a^2\left(1-4\frac{x}{a}+6\frac{x^2}{a^2}-4\frac{x^3}{a^3}+\frac{x^4}{a^4}\right)\left[1-2\frac{x}{a}+3\frac{x^2}{a^2}+\ldots(-1)^r(r+1)\frac{x^r}{a^r}+\ldots\right].$$

Hence the coefficient of x^r

$$= (-1)^r a^{2-r}\{r+1+4r+6(r-1)+4(r-2)+r-3\} = 16(-1)^r(r-1)\,a^{2-r}.$$

9. $(1+x^2)^2(1-x^3)^{-2} = (1+3x^2+3x^4+x^6)\{1+2x^3+3x^6+\ldots$

$$+(n-1)x^{3n-6}+nx^{3n-3}+(n+1)x^{3n}+\ldots\}.$$

The coefficient of x^{3n} is therefore $(n+1)+(n-1) = 2n$.

10. $(1+2x)^3(1-x)^{-2} = (1+6x+12x^2+8x^3)\{1+2x+3x^2+\ldots$

$$\ldots+(n+1)x^n+\ldots\}.$$

Hence the coefficient of x^n, when $n \nless 3$, is

$$(n+1)+6n+12(n-1)+8(n-2) = 27(n-1).$$

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1. (i) $(1-x)^{-\frac{1}{2}} = 1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots;$

$$\therefore \left(1-\frac{3}{4}\right)^{-\frac{1}{2}} = 1+\frac{1}{\lfloor 1}\frac{3}{2^3}+\frac{1\cdot 3}{\lfloor 2}\frac{3^2}{2^6}+\frac{1\cdot 3\cdot 5}{\lfloor 3}\frac{3^3}{2^9}+\ldots$$

(ii) $(1+x)^{-\frac{1}{2}} = 1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots;$

$$\therefore \left(1+\frac{1}{2}\right)^{-\frac{1}{2}} = 1-\frac{1}{2}\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{2^2}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{2^3}+\ldots$$

(iii) $(1-x)^{-\frac{4}{3}}=1+\frac{4}{3}x+\frac{4.7}{3.6}x^2+\frac{4.7.10}{3.6.9}x^3+\ldots;$

$\therefore\left(1-\frac{3}{4}\right)^{-\frac{4}{3}}=1+\frac{4}{\lfloor 1}\frac{1}{4}+\frac{4.7}{\lfloor 2}\frac{1}{4^2}+\frac{4.7.10}{\lfloor 3}\frac{1}{4^3}+\ldots$

(iv) $(1-x)^{-\frac{3}{2}}=1+\frac{3}{2}x+\frac{3.5}{2.4}x^2+\frac{3.5.7}{2.4.6}x^3+\ldots;$

$\therefore\left(1-\frac{2}{3}\right)^{-\frac{3}{2}}=1+\frac{3}{3}+\frac{3.5}{3.6}+\frac{3.5.7}{3.6.9}+\ldots$

(v) $S=\frac{3}{1.2}\frac{1}{2^2}+\frac{3.4}{1.2.3}\frac{1}{2^3}+\frac{3.4.5}{1.2.3.4}\frac{1}{2^4}+\ldots=\frac{1}{2}\left\{3\cdot\frac{1}{2^2}+4\cdot\frac{1}{2^3}+\ldots\right\}.$

Now $\left(1-\frac{1}{2}\right)^{-2}=1+1+3\cdot\frac{1}{2^2}+4\cdot\frac{1}{2^3}+\ldots.$ Hence $S=1.$

(vi) $(1-x)^{-\frac{2}{3}}=1+\frac{2}{3}x+\frac{2.5}{3.6}x^2+\frac{2.5.8}{3.6.9}x^3+\ldots;$

$\therefore\left(1-\frac{1}{2}\right)^{-\frac{2}{3}}=1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\ldots.$

(vii) $(1+x)^{-\frac{3}{2}}=1-\frac{3}{2}x+\frac{3.5}{2.4}x^2-\frac{3.5.7}{2.4.6}x^3+\ldots;$

$\therefore\left(1+\frac{1}{2}\right)^{-\frac{3}{2}}=1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+\ldots$

(viii) $S=\frac{1}{\lfloor 2}\frac{4}{9}+\frac{1.3}{\lfloor 3}\left(\frac{4}{9}\right)^2+\frac{1.3.5}{\lfloor 4}\left(\frac{4}{9}\right)^3+\ldots$

Now $(1-x)^{\frac{1}{2}}=1-\frac{1}{1}\frac{x}{2}-\frac{1.1}{\lfloor 2}\left(\frac{x}{2}\right)^2-\frac{1.1.3}{\lfloor 3}\left(\frac{x}{2}\right)^3-\ldots;$

$\therefore\left(1-\frac{8}{9}\right)^{\frac{1}{2}}=1-\frac{4}{9}-\frac{1}{\lfloor 2}\left(\frac{4}{9}\right)^2-\frac{1.3}{\lfloor 3}\left(\frac{4}{9}\right)^3-\ldots,$

that is $\frac{1}{3}=1-\frac{4}{9}-\frac{4}{9}S,$ whence $S=\frac{1}{2}.$

(ix) $(1-x)^{-\frac{2}{3}}=1+\frac{2}{3}x+\frac{2.5}{3.6}x^2+\ldots;\ \therefore S=\left(1-\frac{1}{3}\right)^{-\frac{2}{3}}.$

(x) $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1.1}{2.4}x^2+\frac{1.1.3}{2.4.6}x^3-\frac{1.1.3.5}{2.4.6.8}x^4+\ldots,$

$\therefore\left(1+\frac{2}{9}\right)^{\frac{1}{2}}=1+\frac{1}{9}-S.$

(xi) $(1-x)^{\frac{3}{2}}=1-\frac{3}{2}x+\frac{3.1}{2.4}x^2+\frac{3.1.1}{2.4.6}x^3+\frac{3.1.1.3}{2.4.6.8}x^4+\ldots$

Hence $(1-1)^{\frac{3}{2}}=1-\frac{3}{2}+\frac{3}{8}+3S\,;\ \therefore\ S=\frac{1}{24}$.

(xii) $S=\frac{1}{3}\cdot\frac{7}{24}+\frac{1.4}{3.4}\left(\frac{7}{24}\right)^2+\frac{1.4.7}{3.4.5}\left(\frac{7}{24}\right)^3+\ldots$

Now $\left(1-\frac{7}{8}\right)^{\frac{5}{3}}=1-\frac{5}{\underline{1}}\cdot\frac{7}{24}+\frac{5.2}{\underline{2}}\left(\frac{7}{24}\right)^2+\frac{5.2.1}{\underline{3}}\left(\frac{7}{24}\right)^3$

$$+\frac{5.2.1.4}{\underline{4}}\left(\frac{7}{24}\right)^4+\frac{5.2.1.4.7}{\underline{5}}\left(\frac{7}{24}\right)^5+\ldots.$$

Hence $\frac{1}{32}=1-\frac{35}{24}+\frac{5.7^2}{24^2}+5\left(\frac{7}{24}\right)^2 S$, whence $S=\frac{37}{245}$.

2. $1+n\frac{a}{a+b}+\frac{n(n+1)}{1.2}\left(\frac{a}{a+b}\right)^2+\ldots=\left(1-\frac{a}{a+b}\right)^{-n}$,

and $1+n\frac{b}{a+b}+\frac{n(n+1)}{1.2}\left(\frac{b}{a+b}\right)^2+\ldots=\left(1-\frac{b}{a+b}\right)^{-n}$;

hence the fraction on the left

$$=\left(\frac{a+b}{b}\right)^n\Big/\left(\frac{a+b}{a}\right)^n=\frac{a^n}{b^n}.$$

3. $(1+x)^n=2^n\left(1+\frac{1-x}{1+x}\right)^{-n}=2^n\left\{1-n\frac{1-x}{1+x}+\frac{n.n+1}{1.2}\left(\frac{1-x}{1+x}\right)^2-\ldots\right\}$.

4. $\frac{x}{1+x}+\frac{1}{2}\left(\frac{x}{1+x}\right)^2+\frac{1.3}{2.4}\left(\frac{x}{1+x}\right)^3+\ldots$

$$=\frac{x}{1+x}\left(1-\frac{x}{1+x}\right)^{-\frac{1}{2}}=\frac{x}{\sqrt{(1+x)}}.$$

The series is not convergent unless the absolute value of $\frac{x}{1+x}$ be less than 1, or $x>-\frac{1}{2}$.

5. $(1-x^2)^n=(1+x)^n\{(1+x)-2x\}^n=(1+x)^{2n}-2nx(1+x)^{2n-1}$

$$+\frac{2n(2n-2)}{1.2}x^2(1+x)^{2n-2}-\ldots\ldots$$

6. $1+n\frac{a-x}{a+x}+\frac{n(n+1)}{1.2}\left(\frac{a-x}{a+x}\right)^2+\ldots=\left(1-\frac{a-x}{a+x}\right)^{-n}=\left(\frac{a+x}{2x}\right)^n.$

7. $(1+x)^{2n}=(1+x)^n\left(1-\frac{x}{1+x}\right)^{-n}$

$=(1+x)^n\left\{1+\frac{nx}{1+x}+\frac{n(n+1)}{1.2}\frac{x^2}{(1+x)^2}+\ldots\right\}=(1+x)^n+nx(1+x)^{n-1}+\ldots$

8. Right side $=(a+b)^4\frac{a^2}{b^2}\left(1+\frac{a}{b}\right)^{-4}=a^2b^2.$

9. We have $\left\{1-n\frac{1}{1+x}+\frac{n(n-1)}{1.2}\frac{1}{(1+x)^2}-\ldots\right\}$

$-\frac{x}{1+x}\left\{1-\frac{n-1}{1}\frac{1}{1+x}+\frac{(n-1)(n-2)}{1.2}\frac{1}{(1+x)^2}-\ldots\right\}$

$=\left(1-\frac{1}{1+x}\right)^n-\frac{x}{1+x}\left(1-\frac{1}{1+x}\right)^{n-1}=0.$

10. $1+n\frac{2y}{x+y}+\frac{n(n+1)}{1.2}\left(\frac{2y}{x+y}\right)^2+\ldots=\left(1-\frac{2y}{x+y}\right)^{-n}$

$=\left(\frac{x-y}{x+y}\right)^{-n}=\left(1+\frac{2y}{x-y}\right)^n=1+n\frac{2y}{x-y}+\frac{n(n-1)}{1.2}\left(\frac{2y}{x-y}\right)^2+\ldots$

In order that both expansions may be convergent the absolute values of $\frac{2y}{x+y}$ and $\frac{2y}{x-y}$ must both be less than unity, and these conditions are satisfied if $(3y)^2<x^2$.

11. $(1-x)^n=1-nx+\frac{n(n-1)}{1.2}x^2-\ldots+(-1)^r\frac{(n)_r}{\lfloor r}x^r+\ldots,$

$(1-x)^{-2}=1+2x+3x^2+\ldots+rx^{r-1}+(r+1)x^r+\ldots$

Hence $r-(r-1)n+(r-2)\frac{n(n-1)}{1.2}-\ldots$ to r terms = coefficient of x^{r-1} in $(1-x)^n\times(1-x)^{-2}$, that is in $(1-x)^{n-2}=(-1)^{r-1}\frac{(n-2)_{r-1}}{\lfloor r-1}$.

12. $(1-x)^{-n}=1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\ldots.$

Also $(1-x)^n=(-1)^n\left\{x^n-nx^{n-1}+\frac{n(n-1)}{1.2}x^{n-2}-\ldots\right\}.$

Hence $(-1)^{n+1}\left\{n-\frac{n^2(n-1)}{\lfloor 1\,\lfloor 2}+\frac{n^2(n^2-1^2)(n-2)}{\lfloor 2\,\lfloor 3}-\ldots\right\}$ is equal to the coefficient of x^{n-1} in $(1-x)^n\times(1-x)^{-n}$. Hence the sum of the given series is zero.

13. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \ldots$

Also $(1-x)^{n-1} = (-1)^{n-1}\left\{x^{n-1} - \frac{n-1}{1}x^{n-2} + \frac{(n-1)(n-2)}{\lfloor 2}x^{n-3} - \ldots\right\}$.

Hence $(-1)^{n-1}\left\{n - \frac{n(n^2-1^2)}{\lfloor 1 \lfloor 2} + \frac{n(n^2-1^2)(n^2-2^2)}{\lfloor 2 \lfloor 3} - \ldots\right\}$ = coefficient of x^n in $(1-x)^{-1}$, and this is **1.**

Hence $n - \frac{n(n^2-1^2)}{\lfloor 1 \lfloor 2} + \ldots = (-1)^{n-1}$.

14. $(1-x)^n = (-1)^n\left\{x^n - nx^{n-1} + \frac{n(n-1)}{1.2}x^{n-2} - \frac{n(n-1)(n-2)}{1.2.3}x^{n-3} + \ldots\right\}$

$$(1-x)^{-4} = 1 + 4x + \frac{4.5}{1.2}x^2 + \frac{4.5.6}{1.2.3}x^3 + \ldots$$

Hence $(-1)^n\left\{1 - 4n + \frac{4.5}{1.2}\frac{n(n-1)}{1.2} - \frac{4.5.6}{1.2.3}\frac{n(n-1)(n-2)}{1.2.3} + \ldots\right\}$ = coefficient of x^n in $(1-x)^{n-4}$, and this is zero if $n \neq 4$.

15. $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots + (n+1)x^n + \ldots$

$$(1-x)^{-3} = \frac{1}{1.2}\{1.2 + 2.3x + 3.4x^2 + \ldots + (n+1)(n+2)x^n + \ldots\}.$$

Hence $\frac{1}{2}\{n(n+1) + 2(n-1)n + 3(n-2)(n-1) + \ldots + n.1.2\}$ = coefficient of x^{n-1} in $(1-x)^{-5} = \frac{1}{\lfloor 4}n(n+1)(n+2)(n+3)$.

16. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \ldots$

Also $2(1-x)^{-3} = 1.2 + 2.3x + 3.4x^2 + \ldots + (n+1)(n+2)x^n + \ldots$

Hence $1.n(n+1) + \frac{n}{1}.(n-1)n + \frac{n(n+1)}{1.2}(n-2)(n-1) + \ldots$

$$= \text{coefficient of } x^{n-1} \text{ in } 2(1-x)^{-n-3}$$

$$= 2\frac{(n+3)(n+4)\ldots(2n+1)}{1.2\ldots(n-1)} = \frac{2\lfloor 2n+1}{\lfloor n+2 \lfloor n-1}.$$

17. $(1-x)^{-\frac{1}{2}} = 1 + p_1x + p_2x^2 + \ldots + p_nx^n + \ldots$

$$(1-x)^{-\frac{1}{2}} = 1 + p_1x + p_2x^2 + \ldots + p_nx^n + \ldots.$$

Hence $p_n.1 + p_{n-1}p_1 + p_{n-2}p_2 + \ldots + 1.p_n$ = coefficient of x^n in $(1-x)^{-1} = 1$.

18. $(1-x)^{-\frac{1}{2}} = 1 + p_1 x + p_2 x^2 + \ldots + p_n x^n + \ldots$

$(1-x)^{-\frac{5}{2}} = 1 + q_1 x + q_2 x^2 + \ldots + q_n x^n + \ldots$.

Hence $p_r \cdot 1 + p_{r-1} q_1 + p_{r-2} q_2 + \ldots + 1 \cdot q_r =$ coefficient of x^r in $(1-x)^{-\frac{1}{2}} \times (1-x)^{-\frac{5}{2}}$, that is in $(1-x)^{-3}$, and this coefficient $= \frac{1}{2}(r+1)(r+2)$.

19. $$\frac{1}{1-x-2x^2} = \frac{1}{3}\left(\frac{1}{1+x} + \frac{2}{1-2x}\right).$$

Also $$\frac{1}{1-x-2x^2} = (1-x)^{-1}\left(1 - \frac{2x^2}{1-x}\right)^{-1}$$

$$= (1-x)^{-1} + 2x^2(1-x)^{-2} + 2^2x^4(1-x)^{-3} + 2^3x^6(1-x)^{-4} + \ldots .$$

The coefficient of x^n in the series on the right is easily seen to be

$$1 + 2(n-1) + 2^2\frac{(n-3)(n-2)}{1.2} + 2^3\frac{(n-5)(n-4)(n-3)}{\lfloor 3} + \ldots .$$

Also the coefficient of x^n in $\frac{1}{3}\left(\frac{1}{1+x} + \frac{2}{1-2x}\right)$ is $\frac{1}{3}\{2^{n+1} + (-1)^n\}$.

20. $$\frac{1}{1-(a+b)x+abx^2} = \frac{1}{a-b}\left(\frac{a}{1-ax} - \frac{b}{1-bx}\right).$$

The coefficient of x^{n-1} in the expansion of the expression on the right is

$$(a^n - b^n)/(a-b).$$

Also $\{1-(a+b)x+abx^2\}^{-1} = \{1-(a+b)x\}^{-1}\left\{1 + \frac{abx^2}{1-(a+b)x}\right\}^{-1}$

$= \{1-(a+b)x\}^{-1} - abx^2\{1-(a+b)x\}^{-2} + a^2b^2x^4\{1-(a+b)x\}^{-3} - \ldots$,

and the coefficient of x^{n-1} in the last expression is

$$(a+b)^{n-1} - (n-2)ab(a+b)^{n-3} + \frac{(n-4)(n-3)}{1.2}a^2b^2(a+b)^{n-5} - \ldots .$$

21. $$(1+2x+x^2)^{2n} = \left(1 + \frac{2x}{1+x^2}\right)^{2n}(1+x^2)^{2n}$$

$$= 2^{2n}x^{2n} + \frac{2n}{\lfloor 1}2^{2n-1}x^{2n-1}(1+x^2) + \frac{2n(2n-1)}{\lfloor 2}2^{2n-2}x^{2n-2}(1+x^2)^2 + \ldots + (1+x^2)^{2n}.$$

The coefficient of x^{2n} in the last expression is easily seen to be

$$2^{2n} + \frac{2n(2n-1)}{\lfloor 2 \lfloor 1}2^{2n-2} \cdot \frac{\lfloor 2}{\lfloor 1 \lfloor 1} + \frac{2n(2n-1)(2n-2)(2n-3)}{\lfloor 4}2^{2n-4} \cdot \frac{\lfloor 4}{\lfloor 2 \lfloor 2}$$

$$+ \ldots + \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

But $$(1+2x+x^2)^{2n}=(1+x)^{4n},$$

and the coefficient of x^{2n} in the expansion of $(1+x)^{4n}$ is

$$\frac{\lfloor 4n}{\lfloor 2n \lfloor 2n}.$$

22. The series on the left is equal to the coefficient of x^n in

$$(1-x)^{-n}\left\{1-nx^3+\frac{n(n-1)}{1.2}x^6+\ldots\right\},$$

that is in $$(1-x)^{-n}(1-x^3)^n, \text{ or in } (1+x+x^2)^n.$$

Now, if $n>2n$, the coefficient of x^n in $(1+x+x^2)^n$ is zero, and the coefficient of x^{2n} is clearly 1.

23. $\{(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\}(1-x)=1-x^\infty=1$, if $x<1$.

Hence $$(1+x)(1+x^2)(1+x^4)\ldots=\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots.$$

24. $$\{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\ldots\}$$
$$\times\{(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots\}$$

is equivalent to the continued product of the following

$$(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots,$$
$$(1-x^3)(1+x^3)(1+x^6)(1+x^{12})(1+x^{24})\ldots,$$
$$(1-x^5)(1+x^5)(1+x^{10})(1+x^{20})(1+x^{40})\ldots, \text{ \&c.}$$

For in the above arrangement every possible factor of the form $1+x^r$ occurs, and occurs only *once*, since every number can be expressed in the form $(2m+1)2^n$ (m and n being positive integers including zero), and can be so expressed in only one way.

Now, when $x<1$, each of the expressions last written is equal to 1, and therefore

$$\{(1-x)(1-x^3)(1-x^5)\ldots\}\{(1+x)(1+x^2)(1+x^3)\ldots\}=1.$$

25. The number required is the coefficient of x^{12} in $(x+x^2+x^3)^6$, that is in $x^6(1-x^3)^6(1-x)^{-6}$

$$=x^6(1-6x^3+15x^6-\ldots)\left(1+\ldots+\frac{6.7.8}{1.2.3}x^3+\ldots+\frac{6.7.8.9.10.11}{\lfloor 6}x^6+\ldots\right).$$

Thus the number required is

$$15-6.7.8+\frac{6.7.8.9.10.11}{\lfloor 6}=141.$$

26. From Art. 288 the number required is the sum of the coefficients of the different powers of x in

$$(1+x+x^2+\ldots x^p)\,(1+x)^{n-p};$$

and this sum is found by putting $x=1$ and subtracting the coefficient of x^0: the sum is therefore $(p+1)\,2^{n-p}-1$.

Or thus:—

We can take 0, or 1, or 2, ..., or p of the p like things, and 0 or 1 of each of the other $n-p$ things. Hence the total number of ways of making a choice is $(p+1)\,2^{n-p}-1$, rejecting the case in which nothing is taken.

27. The number required is the coefficient of x^n in $(1+x+x^2+\ldots+x^n)^r$, that is in $(1-x^{n+1})^r\,.\,(1-x)^{-r}$; and this coefficient is

$$\frac{r\,(r+1)\ldots(r+n-1)}{\lfloor n}={}_{n+r-1}C_{r-1}.$$

28. The number required is the coefficient of x^n in

$$(1+x+x^2+\ldots+x^n)\,(1+x)^n,$$

which is clearly equal to the sum of all the coefficients in the expansion of

$$(1+x)^n=2^n.$$

29. The number required is the coefficient of x^n in

$$(1+x+x^2+\ldots+x^n)\,(1+x)^{2n},$$

which is equal to the sum of the first $n+1$ coefficients in the expansion of

$$(1+x)^{2n}=\frac{1}{2}\{2^{2n}+\text{coefficient of } x^n\}=2^{2n-1}+\frac{1}{2}\,\frac{\lfloor 2n}{\lfloor n\,\lfloor n}.$$

30. The number required is the coefficient of x^{2m} in

$$(1+x+x^2+\ldots+x^m)^4,$$

that is in

$$(1-x^{m+1})^4\,(1-x)^{-4}$$

$$=\frac{1}{6}(1-4x^{m+1}+\ldots)\{1\,.\,2\,.\,3+\ldots+m\,(m+1)\,(m+2)\,x^{m-1}+\ldots$$

$$+(2m+1)\,(2m+2)\,(2m+3)\,x^{2m}+\ldots\}.$$

Hence the number required

$$=\frac{1}{6}\{(2m+1)\,(2m+2)\,(2m+3)-4m\,(m+1)\,(m+2)\}$$

$$=\frac{1}{3}\,(m+1)\,(2m^2+4m+3).$$

31. $(1-2x-2x^2)^{\frac{7}{2}} = 1 - 7x(1+x) + \frac{7 \cdot 5}{1 \cdot 2} x^2 (1+x)^2 - \frac{7 \cdot 5 \cdot 3}{1 \cdot 2 \cdot 3} x^3 (1+x)^3$

$$+ \frac{7 \cdot 5 \cdot 3 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} x^4 (1+x)^4 + \ldots$$

$$= 1 + \ldots + x^4 \left\{ \frac{7 \cdot 5}{1 \cdot 2} - \frac{7 \cdot 5 \cdot 3}{1 \cdot 2 \cdot 3} \cdot 3 + \frac{7 \cdot 5 \cdot 3 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \right\} + \ldots$$

32. $(1+x+x^2+x^3+x^4)^6 = (1-x^5)^6 (1-x)^{-6}$

$$= (1 - 6x^5 + \ldots) \cdot \frac{1}{\lfloor 5} \{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 + \ldots + 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 x^5 + \ldots\}$$

$$= 1 + \ldots + 246x^5 + \ldots$$

Also $(1+x+\ldots+x^5)^8 = (1-x^6)^8 (1-x)^{-8}$

$$= (1 - 8x^6 + \ldots) \cdot \frac{1}{\lfloor 7} \{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 + \ldots + 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 x^5 + \ldots\}$$

$$= 1 + \ldots + 792x^5 + \ldots$$

33. The number required is the coefficient of x^{30} in the expansion of $(x^0+x+x^2+x^3+x^4+x^5)^7$. But since the coefficients of terms equidistant from the beginning and the end are the same the coefficient of x^{30} is the same as the coefficient of x^5; and the coefficient of x^5 in $(1-x^6)^7 (1-x)^{-7}$ is $\frac{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 462$.

34. The number required is the coefficient of x^{20} in the expansion of $(x+x^2+\ldots+x^6)^4$, or the coefficient of x^{16} in the expansion of $(1+x+\ldots+x^5)^4$. The coefficient of x^{16} is equal to the coefficient of x^4, and the coefficient of x^4 in $(1-x^6)^4 (1-x)^{-4}$ is $\frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 35$.

35. $$\frac{1}{4a^2 + 6ax + 9x^2} = \frac{2a - 3x}{8a^3 - 27x^3} = \frac{1}{4a^2} \left(1 - \frac{3x}{2a}\right) \left(1 - \frac{27x^3}{8a^3}\right)^{-1}$$

$$= \frac{1}{4a^2} \left(1 - \frac{3x}{2a}\right) \left\{1 + \left(\frac{3x}{2a}\right)^3 + \left(\frac{3x}{2a}\right)^6 + \ldots + \left(\frac{3x}{2a}\right)^{3n} + \ldots\right\}.$$

Hence the coefficient of x^r is $\frac{1}{4a^2} \left(\frac{3}{2a}\right)^r$ when r is of the form $3n$, and is $-\frac{1}{4a^2} \left(\frac{3}{2a}\right)^r$ when r is of the form $3n+1$, and zero when r is of the form $3n+2$.

36. $$\frac{1+x}{(1+x+x^2)^2} = \frac{(1+x)(1-x)^2}{(1-x^3)^2}$$

$$(1 - x - x^2 + x^3) \{1 + 2x^3 + 3x^6 + \ldots + mx^{3(m-1)} + (m+1) x^{3m} + \ldots\}.$$

Hence the coefficient of x^{3m} is $(m+1)+m$.

37. $(1+2x+3x^2+4x^3+\ldots)^2=\{(1-x)^{-2}\}^2$

$$=(1-x)^{-4}=\frac{1}{\lfloor 3}\{1\,.\,2\,.\,3+2\,.\,3\,.\,4x+\ldots+(r+1)\,(r+2)\,(r+3)\,x^r+\ldots\}.$$

38. $1\,.\,2+2\,.\,3x+3\,.\,4x^2+\ldots=2\,(1-x)^{-3}$. Hence we have $4\,(1-x)^{-6}$, the coefficient of x^n in which is

$$\frac{1}{30}(n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5).$$

39. $\{1\,.\,2+2\,.\,3\,.\,2x+3\,.\,4\,.\,2^2x^2+\ldots\}^n=\{2\,(1-2x)^{-3}\}^n=2^n\,(1-2x)^{-3n}$. Hence the coefficient of x^r is $2^{n+r}\,(3n)\,(3n+1)\,\ldots(3n+r-1)/\lfloor r$.

40. $(1+x+2x^2+3x^3+\ldots)^2=\{1+x\,(1-x)^{-2}\}^2$

$$=1+2x\,(1-x)^{-2}+x^2\,(1-x)^{-4}=1+2x\,(1+2x+3x^2+\ldots+nx^{n-1}+\ldots)$$

$$+\frac{x^2}{6}\{1\,.\,2\,.\,3+2\,.\,3\,.\,4x+\ldots+(n-1)\,n\,(n+1)\,x^{n-2}+\ldots\}$$

Hence the required coefficient

$$=2r+\frac{1}{6}(r-1)\,r\,(r+1)=\frac{1}{6}r\,(r^2+11).$$

41. $$\sqrt[n]{\frac{p}{q}}=\sqrt[n]{(p+q+p-q)}=\frac{\sqrt[n]{\left(1+\dfrac{p-q}{p+q}\right)}}{\sqrt[n]{\left(1+\dfrac{q-p}{p+q}\right)}}=\frac{1+\dfrac{1}{n}\dfrac{p-q}{p+q}}{1+\dfrac{1}{n}\dfrac{q-p}{p+q}},$$

neglecting squares and higher powers of $(p-q)/(p+q)$

$$=\frac{(n+1)\,p+(n-1)\,q}{(n+1)\,q+(n-1)\,p}.$$

42. Since $6\sqrt{6}-14$ is a proper fraction, it follows that $(6\sqrt{6}-14)^{2n+1}$ is a proper fraction.

Now

$$(6\sqrt{6}+14)^{2n+1}-(6\sqrt{6}-14)^{2n+1}=2\left\{14^{2n+1}+\frac{(2n+1)\,2n}{1\,.\,2}14^{2n-1}\,.\,6^3\right.$$

$$\left.+\frac{(2n+1)\,2n\,(2n-1)\,(2n-2)}{1\,.\,2\,.\,3\,.\,4}14^{2n-3}\,.\,6^6+\ldots+\frac{(2n+1)}{1}14\,.\,6^{3n}\right\}.$$

Hence $(6\sqrt{6}+14)^{2n+1}-(6\sqrt{6}-14)^{2n+1}$ is an integer, and therefore, as $(6\sqrt{6}-14)^{2n+1}$ is a proper fraction, $(6\sqrt{6}-14)^{2n+1}$ is the fractional part of N. Hence $NF=(6\sqrt{6}+14)^{2n+1}\,(6\sqrt{6}-14)^{2n+1}=20^{2n+1}$.

43. $(3\sqrt{3}+5)^{2r+1}-(3\sqrt{3}-5)^{2r+1}=2\left\{5^{2r+1}+\frac{(2r+1)\,2r}{1\,.\,2}\,5^{2r-1}3^3+\ldots\right.$

$$\left.+\frac{2r+1}{1}\,5\,.\,3^{3r}\right\}=\text{an integer.}$$

Also $(3\sqrt{3}-5)^{2r+1}$ is a proper fraction, and therefore it is the fractional part of $(3\sqrt{3}+5)^{2r+1}$. Thus

$$F(I+F)=(3\sqrt{3}+5)^{2r+1}(3\sqrt{3}-5)^{2r+1}=2^{2r+1}.$$

44. $(3+\sqrt{7})^{2m}+(3-\sqrt{7})^{2m}=2\left\{3^{2m}+\frac{2m\,(2m-1)}{1\,.\,2}\,3^{2m-2}\,.\,7\right.$

$$\left.+\frac{2m\,(2m-1)\,(2m-2)\,(2m-3)}{\underline{|4}}\,3^{2m-4}\,.\,7^2+\ldots+7^m\right\}=\text{an integer.}$$

Also $(3-\sqrt{7})^{2m}$ is a proper fraction, and therefore $(3+\sqrt{7})^{2m}+(3-\sqrt{7})^{2m}$ is the integer next above $(3+\sqrt{7})^{2m}$.

Putting I_{2r} for $(3+\sqrt{7})^{2r}+(3-\sqrt{7})^{2r}$, we have $I_{2r-2}\,.\,I_2=I_{2r}+2^2\,.\,I_{2r-4}$, that is $I_{2r}=2^5\,.\,I_{2r-1)}-2^2I_{2(r-2)}$. Hence, if $I_{2(r-2)}$ is divisible by 2^{r-2+1} and $I_{2(r-1)}$ by 2^r, then will I_{2r} be divisible by 2^{r+1}. Now it is easily seen that I_2 is divisible by 2^2 and I_4 by 2^3; it therefore follows that I_6 is divisible by 2^4; and so on indefinitely.

45. As in 44, it is easily proved that the integer next greater than $(3+\sqrt{5})^m$ is $(3+\sqrt{5})^m+(3-\sqrt{5})^m$.

Putting I_r for $(3+\sqrt{5})^r+(3-\sqrt{5})^r$, we have

$$I_m=28I_{m-2}-16I_{m-4}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i).}$$

Now $I_1=2\,.\,3$, $I_2=2^2\,.\,7$, $I_3=2^4\,.\,9$ and $I_4=2^4\,.\,47$.

Since I_1 is a multiple of 2^1 and I_3 of 2^3, it follows from (i) that I_5 is a multiple of 2^5; and then that I_7 is a multiple of 2^7; and so on for all odd values of m.

Again, since I_2 is a multiple of 2^2 and I_4 of 2^4, it follows from (i) that I_6 is a multiple of 2^6; and then that I_8 is a multiple of 2^8; and so on for all even values. Thus I_m is a multiple of 2^m for all integral values of m.

46. $\dfrac{1+x+y+xy}{1+x+y}=1+\dfrac{xy}{1+x+y}=1+xy\,\{1-(x+y)+(x+y)^2-\ldots$

$$+(-1)^{m+n-2}(x+y)^{m+n-2}+\ldots\}.$$

The coefficient of x^my^n in the expansion is therefore the coefficient of $x^{m-1}y^{n-1}$ in $(-1)^{m+n-2}(x+y)^{m+n-2}$, and thus is

$$(-1)^{m+n}\,\frac{\underline{|m+n-2}}{\underline{|m-1}\;\underline{|n-1}}.$$

47. $\dfrac{x}{(1-x)^2-cx} = x(1-x)^{-2}[1-cx(1-x)^{-2}]^{-1}$

$= x(1-x)^{-2} + cx^2(1-x)^{-4} + c^2x^3(1-x)^{-6} + \ldots$

The coefficient of x^r in $x(1-x)^{-2}$ is r,

the coefficient of x^r in $cx^2(1-x)^{-4}$ is $\dfrac{c}{\lfloor 3}(r-1)\,r\,(r+1)$,

the coefficient of x^r in $c^2x^3(1-x)^{-6}$ is $\dfrac{c^2}{\lfloor 5}(r-2)\,(r-1)\,r\,(r+1)\,(r+2)$,

and so on. Hence the coefficient of x^r in the expansion is

$$r\left\{1 + c\,\frac{r^2-1}{\lfloor 3} + c^2\,\frac{(r^2-1)\,(r^2-2^2)}{\lfloor 5} + \ldots\right\}.$$

48. $2(1-x)^{-3} = 1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \ldots + (2n-1)\,2n \cdot x^{2n-2} + \ldots,$

and $(1+x^2)^n = x^{2n} + n \cdot x^{2n-2} + \dfrac{n(n-1)}{1 \cdot 2}\,x^{2n-4} + \ldots n \cdot x^2 + 1.$

Hence $1 \cdot 2n + 3 \cdot 4\dfrac{n(n-1)}{1 \cdot 2} + 5 \cdot 6\,\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \ldots + (2n-1)\,2n \cdot 1$

is the coefficient of x^{2n-2} in $2(1+x^2)^n(1-x)^{-3}$.

Now $(1+x^2)^n = [2-(1-x)\{2-(1-x)\}]^n$

$= 2^n - n2^{n-1}(1-x)\{2-(1-x)\} + n(n-1)\,2^{n-3}(1-x)^2\{2-(1-x)\}^2$

+ other terms containing $(1-x)^3$.

Hence $\dfrac{2(1+x^2)^n}{(1-x)^3} = \dfrac{2^{n+1}}{(1-x)^3} - \dfrac{n \cdot 2^{n+1}}{(1-x)^2} + \dfrac{n^2 \cdot 2^n}{1-x}$ + an integral expression of the $(2n-3)$th degree.

Hence the coefficient of x^{2n-2} in the expansion of $2(1+x^2)^n(1-x)^{-3}$ is

$$2^{n+1} \cdot \frac{(2n-1)\,2n}{1 \cdot 2} - n \cdot 2^{n+1}(2n-1) + n^2 \cdot 2^n = 2^n \cdot n^2.$$

49. $\dfrac{(1+x)^n}{(1-x)^2} = \dfrac{\{2-(1-x)\}^n}{(1-x)^2} = \dfrac{2^n}{(1-x)^2} - \dfrac{n2^{n-1}}{1-x}$ + an integral expression of the $(n-2)$th degree.

Hence the coefficient of x^{n+r-1} is $2^n(n+r) - n \cdot 2^{n-1} = 2^{n-1}(n+2r)$.

50. $\dfrac{(1-3x)^n}{(1-2x)^2} = \dfrac{\left\{\dfrac{3}{2}(1-2x) - \dfrac{1}{2}\right\}^n}{(1-2x)^2} = \dfrac{(-1)^n}{2^n(1-2x)^2} + \dfrac{(-1)^{n-1}3n}{2^n(1-2x)}$ + an integral expression of the $(n-2)$th degree.

Hence the coefficient of x^{n+r-1} in the expansion

$$= \frac{(-1)^n}{2^n}\{(n+r)\,2^{n+r-1} - 3n \cdot 2^{n+r-1}\} = (-1)^n 2^{r-1}(r-2n).$$

51. From Ex. 4, Art. 294, we have

$$a^n - n(a+b)^n + \frac{n(n-1)}{1\,.\,2}(a+2b)^n - \ldots = (-b)^n \,\underline{|n}.$$

Put $a=n$ and $b=-2$; then

$$n^n - n(n-2)^n + \frac{n(n-1)}{1\,.\,2}(n-4)^n - \ldots = 2^n \,\underline{|n}.$$

52. From Ex. 4, Art. 294, we have

$$\frac{c_0}{1+ay^{-1}} - \frac{c_1}{1+(a+b)y^{-1}} + \frac{c_2}{1+(a+2b)y^{-1}} - \ldots$$

$$= \frac{\underline{|n}b^n y^{-n}}{\{1+ay^{-1}\}\{1+(a+b)y^{-1}\}\ldots\{1+(a+nb)y^{-1}\}}.$$

The coefficient of y^{-n-1} on the left is

$$(-1)^{n+1}[c_0 a^{n+1} - c_1(a+b)^{n+1} + \ldots].$$

The coefficient of y^{-n-1} in $\dfrac{\underline{|n}b^n y^{-n}}{\{1+ay^{-1}\}\ldots\{1+(a+nb)y^{-1}\}}$,

that is, in $\dfrac{\underline{|n}b^n y^{-n}}{1+y^{-1}\{a+(a+b)+\ldots+(a+nb)\}+\text{higher powers of } y^{-1}}$

is $$-\underline{|n}b^n\{a+(a+b)+\ldots+(a+nb)\}.$$

Hence $$a^{n+1} - n(a+b)^{n+1} + \frac{n(n-1)}{1\,.\,2}(a+2b)^{n+1} - \ldots$$

$$= \frac{1}{2}(-1)^n \,\underline{|n+1}\, b^n (2a+nb).$$

53. Let the three terms be the $(r+1)$th, $(r+2)$th and the $(r+3)$th terms in the expansion of $(1+x)^n$. Then we have

$$p_r + \frac{(n-r)(n-r-1)}{(r+1)(r+2)}p_r = 2\frac{n-r}{r+1}p_r,$$

whence $$(n-r)^2 - (2r+5)(n-r) + \frac{1}{4}(2r+5)^2 = \frac{1}{4}(8r+17).$$

Hence, if n is rational, $8r+17$ must be a square, and therefore an *odd* square $(2q+1)^2$ suppose. Then, if $8r+17=(2q+1)^2$, $r=\frac{1}{2}q(q+1)-2$; and therefore $n-r-\frac{1}{2}(2r+5)=\pm\frac{1}{2}(2q+1)$ gives $n=(q+1)^2-2$ or $n=q^2-2$.

54. Let $(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n}x^{2n}$.

Then [see XXV. 24]

$$(1+x+x^2)^n = a_0 x^{2n} + a_1 x^{2n-1} + a_2 x^{2n-2} + \ldots + a_{2n}.$$

Hence $\Sigma a_r^2 =$ coefficient of x^{2n} in $(1+x+x^2)^{2n}$.

Now $(1+x+x^2)^{2n} = \{1+x(1+x)\}^{2n} = 1+2nx(1+x)+\ldots$

$$+\frac{\lfloor 2n}{\lfloor n \lfloor n} x^n (1+x)^n + \ldots + \frac{\lfloor 2n}{\lfloor n+r \lfloor n-r} x^{n+r}(1+x)^{n+r} + \ldots,$$

and the coefficient of x^{2n} in $x^{n+r}(1+x)^{n+r}$ is $\dfrac{\lfloor n+r}{\lfloor n-r \lfloor 2r}$.

Hence the coefficient of x^{2n} in $(1+x+x^2)^{2n}$ is

$$\Sigma_0^n \frac{\lfloor 2n}{\lfloor n+r \lfloor n-r} \frac{\lfloor n+r}{\lfloor n-r \lfloor 2r}, \text{ that is } \Sigma_0^n \frac{\lfloor 2n}{\lfloor 2r \lfloor n-r} \frac{}{\lfloor n-r},$$

which is the same as $\Sigma_0^n \dfrac{\lfloor 2n}{\lfloor 2n-2r \lfloor r \lfloor r}$.

55. $(1-2x)^{-r} = (1-x)^{-2r}\left\{1-\dfrac{x^2}{(1-x)^2}\right\}^{-r}$

$$= (1-x)^{-2r} + r \cdot x^2 (1-x)^{-2r-2} + \frac{r(r+1)}{1.2} x^4 (1-x)^{-2r-4} + \ldots$$

Equating coefficients of x^n, we have

$$2^n \frac{r(r+1)\ldots(r+n-1)}{\lfloor n} = \frac{2r(2r+1)\ldots(2r+n-1)}{\lfloor n}$$

$$+ r \cdot \frac{(2r+2)(2r+3)\ldots(2r+n-1)}{\lfloor n-2} + \frac{r(r+1)}{1.2} \frac{(2r+4)(2r+5)\ldots(2r+n-1)}{\lfloor n-4} + \ldots$$

Hence, dividing by $\{2r(2r+1)\ldots(2r+n-1)\}/\lfloor n$, we have

$$2^n \frac{r(r+1)\ldots(r+n-1)}{2r(2r+1)\ldots(2r+n-1)} = 1 + \frac{n(n-1)}{2(2r+1)} + \frac{n(n-1)(n-2)(n-3)}{2.4(2r+1)(2r+3)} + \ldots$$

XXIX. Pages 369—372.

1. Assume that $\dfrac{3x}{x^2+7x+6} = \dfrac{A}{x+6} + \dfrac{B}{x+1}$;

then $3x = A(x+1) + B(x+6)$ identically.

Putting $x = -6$ we have $A = \dfrac{18}{5}$;

and putting $x = -1$ we have $B = -\dfrac{3}{5}$.

2. Assume that $\dfrac{x+1}{x^2-5x+6}=\dfrac{A}{x-2}+\dfrac{B}{x-3}$;

then $x+1=A(x-3)+B(x-2)$ identically.

Putting $x=2$ we have $A=-3$,

and putting $x=3$ we have $B=4$.

3. Assume that

$$\frac{x}{(x+1)(x+3)(x+5)}=\frac{A}{x+1}+\frac{B}{x+3}+\frac{C}{x+5};$$

then $x=A(x+3)(x+5)+B(x+5)(x+1)+C(x+1)(x+3)$.

Putting in succession -1, -3 and -5 for x, we have

$$A=-\frac{1}{8},\quad B=\frac{3}{4}\quad\text{and}\quad C=-\frac{5}{8}.$$

4. Assume that

$$\frac{x^2+1}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2};$$

then $x^2+1=A(x+1)^2+Bx(x+1)+Cx$ identically.

Put $x=0$, then we have $A=1$; put $x=-1$, then we have $2=-C$; equate coefficients of x^2, then $1=A+B$, and therefore $B=1-A=0$.

5. Assume

$$\frac{8-x}{(2-x)^2(1+x)}=\frac{A}{1+x}+\frac{B}{2-x}+\frac{C}{(2-x)^2};$$

then $8-x=A(2-x)^2+B(1+x)(2-x)+C(1+x)$.

Put $x=-1$, then we have $A=1$. Put $x=2$, then we have $C=2$. Also, equating coefficient of x^2,

$$0=A-B.$$

6. Assume

$$\frac{x^2+x+1}{x^3-4x^2+x+6}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-3};$$

then $x^2+x+1=A(x-2)(x-3)+B(x-3)(x+1)+C(x+1)(x-2)$.

Then put in succession -1, 3 and 2 for x.

7. Assume $\dfrac{x^2-3}{(x+2)(x^2+1)}=\dfrac{A}{x+2}+\dfrac{Bx+C}{x^2+1}$;

then $x^2-3=A(x^2+1)+(Bx+C)(x+2)$.

Put $x=-2$, then $1=5A$. Equate coefficients of x^2 and of x^0; then

$$1=A+B \text{ and } -3=A+2C,$$

whence $$B=\frac{4}{5} \text{ and } C=-\frac{8}{5}.$$

8. Assume

$$\frac{1+7x-x^2}{(1+3x)^2(1-10x)}=\frac{A}{1-10x}+\frac{B}{1+3x}+\frac{C}{(1+3x)^2}.$$

Hence $1+7x-x^2=A(1+3x)^2+B(1-10x)(1+3x)+C(1-10x)$.

Put $$x=\frac{1}{10}, \text{ then } A=1;$$

put $$x=-\frac{1}{3}, \text{ then } C=-\frac{1}{3};$$

equate coefficient of x^2, then $-1=9A-30B$, whence $B=\frac{1}{3}$.

9. Assume

$$\frac{x^2-x+1}{(x^2+1)(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}.$$

Then $x^2-x+1=A(x-1)(x^2+1)+B(x^2+1)+\{Cx+D\}(x-1)^2$.

Put $x=1$, then $B=\frac{1}{2}$. Substitute this value of B, then

$$(x-1)^2=2A(x-1)(x^2+1)+2(Cx+D)(x-1)^2,$$

therefore $$x-1=2A(x^2+1)+2(Cx+D)(x-1).$$

Now put $x=1$, then $A=0$. Equating coefficients of x^2 we have $C=0$, and then $D=\frac{1}{2}$. Thus the given fraction $=\frac{1}{2(x-1)^2}+\frac{1}{2(x^2+1)}$.

10. Assume

$$\frac{5-9x}{(1-3x)^3(1+x)}=\frac{A}{1+x}+\frac{B}{(1-3x)^3}+\frac{C}{(1-3x)^2}+\frac{D}{1-3x}.$$

Then $5-9x=A(1-3x)^3+B(1+x)+C(1+x)(1-3x)$
$+D(1+x)(1-3x)^2$ identically.

Put $x=-1$, then $A=\frac{7}{32}$. Put $x=\frac{1}{3}$, then $B=\frac{3}{2}$. Equate the coefficients

of x^3 and of x^0, then $-27A+9D=0$ and $5=A+B+C+D$.

Hence $D=3A=\frac{21}{32}$, and then

$$C=5-\frac{7}{32}-\frac{3}{2}-\frac{21}{32}=\frac{21}{8}.$$

11. Assume

$$\frac{6x^2+x-1}{(x^2+1)(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}+\frac{Cx+D}{x^2+1}.$$

Then

$$6x^2+x-1=A(x+3)(x^2+1)+B(x-2)(x^2+1)+(Cx+D)(x-2)(x+3).$$

Put $x=2$, then $A=1$; put $x=-3$, then $B=-1$; equate coefficients of x^3, then $0=A+B+C$, therefore $C=-A-B=-1+1=0$; equate coefficients of x^0, then $-1=3A-2B-6D$, therefore $D=1$.

12. Assume $\frac{x^3+2}{(x-2)^2(x^2+1)}=\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}$;

then $x^3+2=A(x^2+1)+B(x-2)(x^2+1)+(Cx+D)(x-2)^2$.

Put $x=2$, then $A=2$. Hence

$$x^3+2-2(x^2+1)=B(x-2)(x^2+1)+(Cx+D)(x-2)^2;$$

or by dividing by $x-2$,

$$x^2=B(x^2+1)+(Cx+D)(x-2).$$

Now put $x=2$, then $B=\frac{4}{5}$; and equating the coefficients of x^2 and of x^0 we have $1=B+C$ and $0=B-2D$, whence $C=\frac{1}{5}$ and $D=\frac{2}{5}$.

13. Assume $\frac{x^2+x}{(x-1)^2(x^2+4)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$;

then $x^2+x=A(x^2+4)+B(x-1)(x^2+4)+(Cx+D)(x-1)^2$.

Put $x=1$, then $A=\frac{2}{5}$. Hence

$$x^2+x-\frac{2}{5}(x^2+4)=B(x-1)(x^2+4)+(Cx+D)(x-1)^2;$$

$$\therefore\ \frac{1}{5}(3x+8)=B(x^2+4)+(Cx+D)(x-1).$$

Now put $x=1$, then $B=\frac{11}{25}$; and equating the coefficients of x^2 and x^0 we have

$$0=B+C \text{ and } \frac{8}{5}=4B-D,$$

whence

$$C=-\frac{11}{25} \text{ and } D=\frac{4}{25}.$$

14. Assume

$$\frac{x^2-x+1}{(x-1)^2(x-2)(x^2+1)}=\frac{A}{x-2}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{Dx+E}{x^2+1}.$$

Then $x^2-x+1=A(x-1)^2(x^2+1)+B(x-2)(x^2+1)$
$+C(x-1)(x-2)(x^2+1)+(Dx+E)(x-2)(x-1)^2.$

Put $x=1$, then $B=-\frac{1}{2}$; put $x=2$, then $A=\frac{3}{5}$.

Put $x=i$, then $-i=(Di+E)(i-2)(-2i)$;

therefore $1=2\{-D+(E-2D)i-2E\}$,

and equating the real and imaginary parts we have

$$1=-2D-4E \text{ and } E-2D=0, \text{ whence } D=-\frac{1}{10},\ E=-\frac{1}{5}.$$

Equate the coefficients of x^4, then $0=A+C+D$, therefore

$$C=-\frac{3}{5}+\frac{1}{10}=-\frac{1}{2}.$$

15. Assume

$$\frac{1+2x}{x^2(x+2)^3(x+1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+1}+\frac{D}{(x+2)^3}+\frac{E}{(x+2)^2}+\frac{F}{x+2}.$$

Then $1+2x=A(x+2)^3(x+1)+Bx(x+2)^3(x+1)+Cx^2(x+2)^3$
$+Dx^2(x+1)+Ex^2(x+1)(x+2)+Fx^2(x+1)(x+2)^2.$

Put $0, -1, -2$ in succession for x; then

$$A=\frac{1}{8},\ C=-1 \text{ and } D=\frac{3}{4}.$$

Substitute these values, then we have

$$\begin{aligned}&8Bx(x+1)(x+2)^3+8Ex^2(x+1)(x+2)+8Fx^2(x+1)(x+2)^2\\&=8+16x-(x+1)(x+2)^3+8x^2(x+2)^3-6x^2(x+1)\\&=8x^5+47x^4+83x^3+40x^2-4x\\&=x(x+1)(x+2)(8x^2+23x-2).\end{aligned}$$

Hence $$8B(x+2)^2+8Ex+8Fx(x+2)=8x^2+23x-2.$$

Now put $x=0$ and we have $B=-\frac{1}{16}$; put $x=-2$ and we have $E=1$; also $8B+8F=8$, whence $F=\frac{17}{16}$.

16. Assume

$$\frac{1+2x}{x^2(x+2)^3(x-1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{(x+2)^3}+\frac{D}{(x+2)^2}+\frac{E}{x+2}+\frac{F}{x-1}.$$

Then $1+2x=A(x+2)^3(x-1)+Bx(x-1)(x+2)^3+Cx^2(x-1)$
$+Dx^2(x-1)(x+2)+Ex^2(x-1)(x+2)^2+Fx^2(x+2)^3.$

Put 0, 1, -2 in succession for x; then

$$A=-\frac{1}{8},\ F=\frac{1}{9}\text{ and }C=\frac{1}{4}.$$

Substitute these values; then we have

$$72Bx(x-1)(x+2)^3+72Dx^2(x-1)(x+2)+72Ex^2(x-1)(x+2)^2$$
$$=72+144x+9(x-1)(x+2)^3-18x^2(x-1)-8x^2(x+2)^3$$
$$=-8x^5-39x^4-69x^3+8x^2+108x$$
$$=-x(x-1)(x+2)(8x^2+31x+54).$$

Hence $72B(x+2)^2+72Dx+72Ex(x+2)=-8x^2-31x-54.$

Now put $x=0$ and we have $B=-\frac{3}{16}$; put $x=-2$ and we have $D=\frac{1}{6}$; also $72B+72E=-8$, whence $E=\frac{11}{144}$.

17. $$\frac{x+4}{x^2+5x+6}=\frac{2}{x+2}-\frac{1}{x+3}=\left(1+\frac{x}{2}\right)^{-1}-\frac{1}{3}\left(1+\frac{x}{3}\right)^{-1}$$
$$=\ldots+(-1)^n(2^{-n}-3^{-n-1})x^n+\ldots$$

18. $$\frac{x-2}{(x+2)(x-1)^2}=\frac{4}{9(x-1)}-\frac{1}{3(x-1)^2}-\frac{4}{9(x+2)}$$
$$=-\frac{4}{9}(1-x)^{-1}-\frac{1}{3}(1-x)^{-2}-\frac{2}{9}\left(1+\frac{x}{2}\right)^{-2},$$

in which the coefficient of x^n is $-\frac{4}{9}-\frac{1}{3}(n+1)-\frac{2}{9}(-1)^n2^{-n}$.

19. $$\frac{x+5}{(x^2-1)(x+2)}=\frac{1}{x-1}-\frac{2}{x+1}+\frac{1}{x+2}$$
$$=-(1-x)^{-1}-2(1+x)^{-1}+\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1},$$

in which the coefficient of x^{2n-1} is $-1-2(-1)^{2n-1}+(-1)^{2n-1}2^{-2n}=1-2^{-2n}$.

20. The sum required is the coefficient of x^{n-1} in the expansion of

$$\frac{3-2x}{(1-2x-3x^2)(1-x)} \text{ [Art. 287].}$$

Now $$\frac{3-2x}{(1-2x-3x^2)(1-x)} = \frac{21}{8(1-3x)} - \frac{1}{4(1-x)} + \frac{5}{8(1+x)},$$

in which the coefficient of x^{n-1} is $\frac{7}{8}3^n - \frac{1}{4} + \frac{5}{8}(-1)^{n-1}$.

21. The sum required is the coefficient of x^{n-1} in the expansion of

$$\frac{2-5x}{(1-5x)(1-3x)(1-2x)(1-x)} \text{ [Art. 287].}$$

Now $$\frac{2-5x}{(1-5x)(1-3x)(1-2x)(1-x)}$$

$$= \frac{125}{24(1-5x)} - \frac{9}{4(1-3x)} - \frac{4}{3(1-2x)} + \frac{3}{8(1-x)},$$

in which the coefficient of x^{n-1} is $\frac{1}{24}5^{n+2} - \frac{1}{4}3^{n+1} - \frac{1}{3}2^{n+1} + \frac{3}{8}$.

22. $\frac{(1+x)^n}{(1-x)^3} = \frac{\{2-(1-x)\}^n}{(1-x)^3} = \frac{2^n}{(1-x)^3} - \frac{n2^{n-1}}{(1-x)^2} + \frac{n(n-1)2^{n-3}}{1-x} +$ an integral expression of the $(n-3)$th degree.

Hence the coefficient of x^n is

$$2^{n-1}(n+1)(n+2) - n2^{n-1}(n+1) + n(n-1)2^{n-3} = 2^{n-3}(n^2+7n+8).$$

The sum of the first n coefficients is the coefficient of x^{n-1} in $\frac{(1+x)^n}{(1-x)^4}$.

Now $(1+x)^n(1-x)^{-4} = \{2-(1-x)\}^n(1-x)^{-4} = 2^n(1-x)^{-4} - n2^{n-1}(1-x)^{-3}$

$+ n(n-1)2^{n-3}(1-x)^{-2} - \frac{1}{3}n(n-1)(n-2)2^{n-4}(1-x)^{-1}$

+ an integral expression of the $(n-4)$th degree.

The coefficient of x^{n-1} is therefore

$$2^n \frac{n(n+1)(n+2)}{1.2.3} - n2^{n-1} \cdot \frac{n(n+1)}{1.2}$$

$$+ n(n-1)2^{n-3} \cdot n - \frac{1}{3}n(n-1)(n-2)2^{n-4}$$

23. $\dfrac{(1+3x)^n}{(1+2x)^2} = \dfrac{\left\{-\frac{1}{2}+\frac{3}{2}(1+2x)\right\}^n}{(1+2x)^2} = \left(-\frac{1}{2}\right)^n \{(1+2x)^{-2} - 3n\,(1+2x)^{-1}$

+ an integral expression of the $(n-2)$th degree$\}$.

Hence the required coefficient

$$= \left(-\frac{1}{2}\right)^n \{(n+r+1)(-2)^{n+r} - 3n(-2)^{n+r}\} = (-2)^r (r+1-2n).$$

24. Since $(x-a_1)(x-a_2)\ldots(x-a_n) = x^n - (a_1+a_2+\ldots+a_n)x^{n-1} +$ terms containing powers of x lower than x^{n-1}, it follows that

$$(x-a_1)(x-a_2)\ldots(x-a_n)\{x+a_1+a_2+\ldots+a_n\} = x^{n+1}$$

+ terms containing powers of x *lower than* x^n. Hence we may assume

$$\frac{x^{n+1}}{(x-a_1)(x-a_2)\ldots(x-a_n)} = x+a_1+a_2+\ldots+a_n + \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n}.$$

Multiply by $\{(x-a_1)(x-a_2)\ldots(x-a_n)\}$ and put $x=a_r$; then

$$a_r^{n+1} = A_r(a_r-a_1)(a_r-a_2)\ldots(a_r-a_n).$$

25. $\{(1-z)(1-cz)(1-c^2z)(1-c^3z)\}^{-1}$ can be expressed in the form

$$A(1-z)^{-1} + B(1-cz)^{-1} + C(1-c^2z)^{-1} + D(1-c^3z)^{-1},$$

and the coefficient of x^{n-1} in the expansion can then be written down.

Or thus:

Let $\dfrac{1}{(1-z)(1-cz)(1-c^2z)(1-c^3z)} = A_0 + A_1z + A_2z^2 + \ldots + A_rz^r + \ldots\ldots$(i).

Change z into cz; then

$$\frac{1}{(1-cz)(1-c^2z)(1-c^3z)(1-c^4z)} = A_0 + A_1cz + A_2c^2z^2 + \ldots + A_rc^rz^r + \ldots$$(ii).

Hence $(1-z)(A_0 + A_1z + A_2z^2 + \ldots + A_rz^r + \ldots)$

$$= (1-c^4z)(A_0 + A_1cz + A_2c^2z^2 + \ldots + A_rc^rz^r + \ldots).$$

Equate the coefficients of z^r on the two sides of the last identity; then

$$A_r - A_{r-1} = A_rc^r - A_{r-1}c^{r+3};$$

$\therefore A_r = \dfrac{1-c^{r+3}}{1-c^r} A_{r-1}$; hence also

$$A_r = \frac{1-c^{r+3}}{1-c^r} \cdot \frac{1-c^{r+2}}{1-c^{r-1}} A_{r-2} = \frac{(1-c^{r+3})(1-c^{r+2})\ldots(1-c^4)}{(1-c^r)(1-c^{r-1})\ldots(1-c)} A_0.$$

But A_0 is obviously 1; we therefore have

$$A_r = \frac{(1-c^{r+3})(1-c^{r+2})(1-c^{r+1})}{(1-c^3)(1-c^2)(1-c)}.$$

26. From Art. 300, Ex. 4, we have

$$H_{m-3}=\Sigma\frac{a^{m+2}}{(a-b)(a-c)(a-a')(a-b')(a-c')}.$$

Take the six terms included in Σ in pairs. One pair of terms is

$$\frac{a^{m+2}}{(a-b)(a-c)(a-a')(a-b')(a-c')}+\frac{a'^{m+2}}{(a'-a)(a'-b)(a'-c)(a'-b')(a'-c')}$$

$$=\frac{a^{m+2}}{(a-a')(a-b)(a-c)\left(a-\frac{cc'}{b}\right)\left(a-\frac{bb'}{c}\right)}$$

$$+\frac{a'^{m+2}}{(a'-a)\left(\frac{cc'}{a}-b\right)\left(\frac{bb'}{a}-c\right)\left(a'-\frac{aa'}{b}\right)\left(a'-\frac{aa'}{c}\right)}$$

$$=\frac{a^2bc\,(a^m-a'^m)}{(a-a')(a-b)(a-c)(ab-cc')(ac-bb')}.$$

Hence $\frac{1}{abc}(b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')\,H_{m-3}$

$$=a(b-c)(bc-aa')(a^m-a'^m)/(a'-a)+\ldots$$

27. From Art. 301, Ex. 1, the coefficient of x^r in the expansion of

$$(1+cx)(1+c^2x)(1+c^3x)\ldots(1+c^nx)$$

is equal to

$$c^{\frac{1}{2}r(r+1)}\frac{(c^n-1)(c^{n-1}-1)\ldots(c^{n-r+1}-1)}{(c^r-1)(c^{r-1}-1)\ldots(c-1)}.$$

Now this coefficient is obviously an *integral* function of c, and therefore $\{(c^n-1)(c^{n-1}-1)\ldots(c^{n-r+1}-1)\}$ is divisible by $\{(c-1)(c^2-1)\ldots(c^r-1)\}$.

28. Let $(1+cx)(1+c^2x)\ldots$ to infinity $=A_0+A_1x+\ldots+A_rx^r+\ldots$

Change x into cx; then, if c be numerically less than unity so that $1+c^nx=1$ when n is very great, we have

$$(1+c^2x)(1+c^3x)\ldots=A_0+A_1cx+\ldots+A_rc^rx^r+\ldots$$

Hence

$$(1+cx)(A_0+A_1cx+\ldots+A_rc^rx^r+\ldots)=A_0+A_1x+\ldots+A_rx^r+\ldots$$

Equating the coefficients of x^r on the two sides of the last identity, we have

$$A_rc^r+A_{r-1}c^r=A_r;$$

$$\therefore\ A_r=\frac{c^r}{1-c^r}A_{r-1},$$

whence we have $$A_r = \frac{c^r}{1-c^r} \cdot \frac{c^{r-1}}{1-c^{r-1}} \cdots \frac{c}{1-c} A_0,$$

and it is obvious that $A_0 = 1$.

29. Let

$$(1+cx)(1+c^3x)(1+c^5x)\ldots\text{to infinity} = A_0 + A_1x + \ldots + A_rx^r + \ldots$$

Change x into c^2x; then, c being less than unity so that $1 + c^{2n+1}x = 1$ when n is very great, we have

$$(1+c^3x)(1+c^5x)\ldots = A_0 + A_1c^2x + \ldots + A_rc^{2r}x^r + \ldots$$

Hence

$$(1+cx)(A_0 + A_1c^2x + \ldots + A_rc^{2r}x^r + \ldots) = (A_0 + A_1x + \ldots + A_rx^r + \ldots).$$

Equating the coefficients of x^r on the two sides of the last identity, we have

$$A_r = A_rc^{2r} + A_{r-1}c^{2r-1}.$$

Hence $$A_r = \frac{c^{2r-1}}{1-c^{2r}} \cdot \frac{c^{2r-3}}{1-c^{2r-2}} \cdots \frac{c}{1-c^2} A_0,$$

and $A_0 = 1$.

30. Let $$\frac{1}{(1-x)(1-cx)(1-c^2x)\ldots} = A_0 + A_1x + \ldots + A_rx^r + \ldots$$

Change x into cx; then

$$\frac{1}{(1-cx)(1-c^2x)(1-c^3x)\ldots} = A_0 + A_1cx + \ldots + A_rc^rx^r + \ldots$$

Hence, as c is less than 1, we have identically

$$A_0 + A_1cx + \ldots + A_rc^rx^r + \ldots = (1-x)(A_0 + A_1x + \ldots + A_rx^r + \ldots).$$

Hence $$A_rc^r = A_r - A_{r-1},$$

whence $$A_r = \frac{1}{1-c^r} \cdot \frac{1}{1-c^{r-1}} \cdots \frac{1}{1-c} A_0,$$

and $A_0 = 1$.

31. Let the given expression $= A_0 + A_1x + \ldots + A_rx^r + \ldots$

Change x into cx; then we have

$$(1+cx)(A_0 + A_1cx + \ldots + A_rc^rx^r + \ldots) = (1-x)(A_0 + A_1x + \ldots + A_rx^r + \ldots).$$

Hence $$A_rc^r + A_{r-1}c^r = A_r - A_{r-1};$$

$$\therefore\ A_r = \frac{1+c^r}{1-c^r} A_{r-1},$$

and therefore $$A_r=\frac{(1+c^r)\,(1+c^{r-1})\ldots(1+c)}{(1-c^r)\,(1-c^{r-1})\ldots(1-c)}A_0.$$

Also A_0 is obviously 1.

32. Proceeding as in 31, we have

$$(1+cx)\,(A_0+A_1cx+\ldots+A_rc^rx^r+\ldots)=(1-cx)\,(A_0+A_1x+\ldots+A_rx^r+\ldots).$$

Hence $$A_rc^r+A_{r-1}c^r=A_r-cA_{r-1};$$

$$\therefore\ A_r=\frac{c\,(c^{r-1}+1)}{1-c^r}A_{r-1},$$

and therefore

$$A_r=\frac{c^{r-1}\,(1+c^{r-1})\,(1+c^{r-2})\ldots\ldots(1+c)}{(1-c^r)\,(1-c^{r-1})\ldots\ldots(1-c^2)}A_1.$$

Also $A_0=1$ and therefore $A_1c+c=A_1-c$, so that $A_1=\frac{c\,(1+1)}{1-c}$.

33. The coefficient of x^r in the expansion of the expression on the left is easily seen to be

$$1+ya^r+y^2a^{2r}+y^3a^{3r}+\ldots\ldots$$

which is equal to $(1-ya^r)^{-1}$: the whole expression on the left is therefore equal to $\Sigma\,\frac{x^r}{1-ya^r}$.

34. The terms on the left are

$$\begin{array}{l} x+x^2+x^3+x^4+x^5+\ldots\ldots \\ 2x^2+2x^4+2x^6+2x^8+2x^{10}+\ldots\ldots \\ 3x^3+3x^6+3x^9+3x^{12}+3x^{15}+\ldots\ldots \\ 4x^4+4x^8+4x^{12}+4x^{16}+4x^{20}+\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Taking them in columns instead of rows, we have

$$\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\frac{x^3}{(1-x^3)^2}+\frac{x^4}{(1-x^4)^2}+\ldots\ldots,$$

35. The terms on the left are

$$\begin{array}{l} x+x^2+x^3+x^4+\ldots\ldots \\ x^2+x^4+x^6+x^8+\ldots\ldots \\ x^3+x^6+x^9+x^{12}+\ldots\ldots \\ x^4+x^8+x^{12}+x^{16}+\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

The top row and the left-hand column together

$$=\frac{x}{1-x}+\frac{x^2}{1-x}=x\,\frac{1+x}{1-x}.$$

Remove the top row and the left-hand column; then the top row and left-hand column of the remainder together

$$= \frac{x^4}{1-x^2} + \frac{x^6}{1-x^2} = x^4 \frac{1+x^2}{1-x^2}.$$

Proceeding in this way we see that the whole expression is equal to

$$= \frac{1+x}{1-x} + x^4 \frac{1+x^2}{1-x^2} + x^9 \frac{1+x^3}{1-x^3} + x^{16} \frac{1+x^4}{1-x^4} + \ldots\ldots$$

XXX. Pages 378—379.

1. $\left(1+\frac{x}{n}\right)^n = \left\{\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right\}^x$. But the limit of $\left(1+\frac{x}{n}\right)^{\frac{n}{x}}$ when n, and therefore also $\frac{n}{x}$, is infinite is e. Hence

$$\left\{\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right\}^x = e^x.$$

2. $$\left(1+\frac{a}{n}\right)^{\frac{n}{b}} = \left\{\left(1+\frac{a}{n}\right)^{\frac{n}{a}}\right\}^{\frac{a}{b}} = e^{\frac{a}{b}},$$

when n, and therefore also $\frac{n}{a}$, is infinite.

3 and 4. $\left(x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\right)^n = e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{1 \cdot 2} e^{(n-2)x} - \ldots$

[Art. 305].

Equate the coefficients of x^{n+1}; then

$$\frac{n}{2} = \frac{n^{n+1}}{\lfloor n+1} - n\frac{(n-1)^{n+1}}{\lfloor n+1} + \frac{n(n-1)}{1 \cdot 2}\frac{(n-2)^{n+1}}{\lfloor n+1} - \ldots;$$

$$\therefore \frac{1}{2} n \lfloor n+1 = n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{1 \cdot 2}(n-2)^{n+1} - \ldots$$

Equate the coefficients of x^{n+2}; then

$$\frac{n}{\lfloor 3} + \frac{n(n-1)}{1 \cdot 2}\frac{1}{(\lfloor 2)^2} = \frac{n^{n+2}}{\lfloor n+2} - n\frac{(n-1)^{n+2}}{\lfloor n+2} + \ldots;$$

$$\therefore \frac{1}{24} n(3n+1) \lfloor n+2 = n^{n+2} - n(n-1)^{n+2} + \ldots$$

5. $$\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\right)=e,$$

and $$\left(1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\ldots\right)=e^{-1}.$$

Hence the product is $e \,.\, e^{-1}=1$.

6. $$e^{-1}=1-\frac{1}{\lfloor 1}+\left(\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}\right)+\left(\frac{1}{\lfloor 4}-\frac{1}{\lfloor 5}\right)+\left(\frac{1}{\lfloor 6}-\frac{1}{\lfloor 7}\right)+\ldots$$
$$=\frac{2}{\lfloor 3}+\frac{4}{\lfloor 5}+\frac{6}{\lfloor 7}+\ldots$$

7. $$2S=2+\frac{2\,.\,3}{\lfloor 2}+\frac{3\,.\,4}{\lfloor 3}+\frac{4\,.\,5}{\lfloor 4}+\frac{5\,.\,6}{\lfloor 5}+\ldots=2+\frac{3}{\lfloor 1}+\frac{4}{\lfloor 2}+\frac{5}{\lfloor 3}+\frac{6}{\lfloor 4}+\ldots$$
$$=\left(2+\frac{2}{\lfloor 1}+\frac{2}{\lfloor 2}+\frac{2}{\lfloor 3}+\frac{2}{\lfloor 4}+\ldots\right)+\left(\frac{1}{\lfloor 1}+\frac{2}{\lfloor 2}+\frac{3}{\lfloor 3}+\frac{4}{\lfloor 4}+\ldots\right)=2e+e.$$

8. $$\left(1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots\right)^2-\left(\frac{1}{\lfloor 1}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 7}+\ldots\right)^2$$
$$=\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\right)\left(1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\ldots\right)=e\,.\,e^{-1}=1.$$

9. $$\frac{1}{1\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,5}+\ldots+\frac{1}{1\,.\,2\,.\,3\ldots(2n-1)\,(2n+1)}+\ldots$$
$$=\left(\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}\right)+\left(\frac{1}{\lfloor 4}-\frac{1}{\lfloor 5}\right)+\ldots+\left(\frac{1}{\lfloor 2n}-\frac{1}{\lfloor 2n+1}\right)+\ldots$$
$$=1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\frac{1}{\lfloor 5}+\ldots=e^{-1}.$$

10. We have to shew that

$$e\left[\frac{1}{\lfloor 1}-\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}-\frac{1}{\lfloor 4}+\ldots\right]=\frac{1}{1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots,$$

or that $e\,(1-e^{-1})=e-1$, which is obvious.

11. $$\frac{e^2+1}{e^2-1}=\frac{e+e^{-1}}{e-e^{-1}}=\frac{2\left(1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\ldots\right)}{2\left(1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\right)}.$$

12. $$1+\frac{1+2x}{\lfloor 1}+\frac{(1+2x)^2}{\lfloor 2}+\frac{(1+2x)^3}{\lfloor 3}+\ldots=e^{1+2x}=e\,.\,e^{2x},$$

and the coefficient of x^n in the expansion of e^{2x} is $2^n/\lfloor n$.

XXXI. Pages 385—388.

1. $$\log(x+n)=\log x\,.\,\frac{x+1}{x}\,.\,\frac{x+2}{x+1}\ldots\ldots\frac{x+n}{x+n-1}$$

$$=\log x+\log\frac{x+1}{x}+\log\frac{x+2}{x+1}+\ldots\ldots+\log\frac{x+n}{x+n-1}.$$

2. $$\log_e\sqrt{12}=\frac{1}{2}\log_e 12=\log_e 3-\frac{1}{2}\log_e\left(1-\frac{1}{4}\right)$$

$$=\left\{\log_e\left(1+\frac{1}{2}\right)-\log_e\left(1-\frac{1}{2}\right)\right\}-\frac{1}{2}\log_e\left(1-\frac{1}{4}\right)=2\left(\frac{1}{2}+\frac{1}{3}\frac{1}{2^3}+\ldots\ldots\right)$$

$$+\frac{1}{2}\left(\frac{1}{4}+\frac{1}{2}\frac{1}{4^2}+\frac{1}{3}\frac{1}{4^3}+\ldots\ldots\right)$$

$$=1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\ldots\ldots$$

3. $$\frac{1}{2}\log_e 10=\frac{3}{2}\log\frac{1+\frac{1}{3}}{1-\frac{1}{3}}+\frac{1}{2}\log_e\frac{1+\frac{1}{9}}{1-\frac{1}{9}}=3\left\{\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\ldots\ldots\right\}$$

$$+\left\{\frac{1}{9}+\frac{1}{3}\frac{1}{9^3}+\frac{1}{5}\frac{1}{9^5}+\ldots\ldots\right\}.$$

4. $$2\log_e 2=1+1-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}-\frac{1}{4}-\frac{1}{4}+\ldots\ldots$$

$$=1+\frac{1}{1\,.\,2}-\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}-\frac{1}{4\,.\,5}+\frac{1}{5\,.\,6}-\frac{1}{6\,.\,7}$$

$$=1+\frac{2}{1\,.\,2\,.\,3}+\frac{2}{3\,.\,4\,.\,5}+\frac{2}{5\,.\,6\,.\,7}+\ldots\ldots,$$

$$\therefore\ \log_e 2-\frac{1}{2}=\frac{1}{1\,.\,2\,.\,3}+\frac{1}{3\,.\,4\,.\,5}+\frac{1}{5\,.\,6\,\;7}+\ldots\ldots$$

5. $$\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\ldots\ldots$$

$$=\frac{3+2}{1.2.3}+\frac{5+2}{3.4.5}+\frac{7+2}{5.6.7}+\ldots\ldots$$

$$=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots+2\left(\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ldots\ldots\right)$$

$$=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+2\left(\log_e 2-\frac{1}{2}\right),\text{ from question 4,}$$

$$=3\log_e 2-1.$$

6. Put $\frac{m}{n}=\frac{x}{x-1}$ and therefore $\frac{m-n}{m+n}=\frac{1}{2x-1}$ in Art. 309, (iv).

7. $$\log_e x=-\log_e\left(1-\frac{x}{x+1}\right)+\log_e\left(1-\frac{1}{x+1}\right)$$

$$=\left\{\frac{x}{x+1}+\frac{1}{2}\frac{x^2}{(x+1)^2}+\ldots\ldots\right\}-\left\{\frac{1}{x+1}+\frac{1}{2}\frac{1}{(x+1)^2}+\ldots\ldots\right\}$$

$$=\frac{x-1}{x+1}+\frac{1}{2}\frac{x^2-1}{(x+1)^2}+\frac{1}{3}\frac{x^3-1}{(x+1)^3}+\ldots\ldots$$

8. $$\log_e\frac{a+x}{a-x}=\frac{1}{2}\log_e\frac{a^2+x^2+2ax}{a^2+x^2-2ax}$$

$$=\frac{1}{2}\log_e\left(1+\frac{2ax}{a^2+x^2}\right)-\frac{1}{2}\log_e\left(1-\frac{2ax}{a^2+x^2}\right)=\frac{2ax}{a^2+x^2}+\frac{1}{3}\left(\frac{2ax}{a^2+x^2}\right)^3+\ldots\ldots$$

9. $$2\log\frac{1+\iota x}{1-\iota x}=\log\frac{1-x^2+2\iota x}{1-x^2-2\iota x}.$$

Hence

$$4\iota\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots\right)=2\iota\left\{\frac{2x}{1-x^2}-\frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3+\frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5+\ldots\ldots\right\}.$$

10. We have [Art. 308]

$$1+xy+\frac{x(x-1)}{1.2}y^2+\frac{x(x-1)(x-2)}{1.2.3}y^3+\ldots\ldots+\frac{x(x-1)\ldots(x-r+1)}{\lfloor r}y^r+\ldots\ldots$$

$$=1+x\log_e(1+y)+\frac{x^2}{\lfloor 2}\{\log_e(1+y)\}^2+\ldots\ldots$$

Equating the coefficients of x^2 on the two sides of this equation, we have

$$\{\log_e(1+y)\}^2 = 2\left\{\frac{y^2}{2} - \frac{1+2}{1.2.3}y^3 + \frac{1.2+2.3+3.1}{\lfloor 4}y^4 + \ldots\ldots\right.$$
$$\left. + (-1)^r \frac{1}{r}\left(\frac{1}{1}+\frac{1}{2}+\ldots\ldots+\frac{1}{r-1}\right)y^r + \ldots\ldots\right\}.$$

11. $\log_e(1+x+x^2) = \log_e(1-x^3) - \log_e(1-x)$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + \frac{x^n}{n} + \ldots\ldots\right) - \left(x^3 + \frac{x^6}{2} + \ldots\ldots + \frac{x^{3n}}{n} + \ldots\ldots\right).$$

Hence if $n = 3m+1$, or if $n = 3m+2$, the coefficient of x^n is $\frac{1}{n}$, but if $n = 3m$ the coefficient of x^n is $\frac{1}{n} - \frac{1}{\frac{n}{3}} = -\frac{2}{n}$.

12. $\log_e(1-x+x^2) = \log_e(1+x^3) - \log_e(1+x)$. Hence

$$a_3 + a_6 + a_9 + \ldots\ldots = \left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{3} - \frac{1}{9}\right) + \ldots\ldots$$
$$= \frac{2}{3}\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots\ldots\right) = \frac{2}{3}\log_e 2.$$

13.
$$\log_e \frac{1+x+x^2}{1-x+x^2} = \log_e \frac{1+x}{1-x} - \log \frac{1+x^3}{1-x^3}$$
$$= 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \ldots\ldots\right\}$$
$$- 2\left\{x^3 + \frac{x^9}{3} + \frac{x^{15}}{5} + \ldots\ldots\right\}$$
$$= 2\left\{x - 2\cdot\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} - 2\frac{x^9}{9} + \ldots\ldots\right\}.$$

14. The coefficient of x^r on the right is easily seen to be

$$= \frac{1}{\lfloor r}\left\{\frac{1}{n} - \frac{\lfloor r}{\lfloor 1 \lfloor r-1}\cdot\frac{1}{n+1} + \frac{\lfloor r}{\lfloor 2 \lfloor r-2}\frac{1}{n+2} + \ldots\ldots + (-1)^r\frac{\lfloor r}{\lfloor r}\frac{1}{n+r}\right\}$$

$$= \frac{1}{\lfloor r}\frac{\lfloor r}{n(n+1)\ldots(n+r)}$$ [Art. 259, Ex. 3]

= coefficient of x^r on the left.

15. Since $2\log_e(1-x)=\log_e\{1-x(2-x)\}$, we have

$$2\Sigma\frac{x^r}{r}=\Sigma\frac{x^r(2-x)^r}{r}.$$

Hence $\frac{2}{n}=$ coefficient of x^n in

$$\frac{x^n(2-x)^n}{n}+\frac{x^{n-1}(2-x)^{n-1}}{n-1}+\frac{x^{n-2}(2-x)^{n-2}}{n-2}+\ldots\ldots$$

$$=\frac{2^n}{n}-2^{n-1}+\frac{(n-3)}{1\,.\,2}2^{n-2}-\frac{(n-4)\,(n-5)}{1\,.\,2\,.\,3}2^{n-3}+\ldots\ldots$$

Hence $2=2^n-n\,.\,2^{n-1}+\frac{n(n-3)}{1\,.\,2}2^{n-1}-\frac{n(n-4)(n-5)}{1\,.\,2\,.\,3}2^{n-3}+\ldots\ldots$

16. $\log_e\frac{1}{1-x-x^2+x^3}=\log_e\frac{1}{(1-x)(1-x^2)}=-\log_e(1-x)-\log_e(1-x^2)$

$$=\left\{x+\frac{x^2}{2}+\ldots+\frac{x^r}{r}+\ldots\right\}+\left\{x^2+\frac{x^4}{2}+\ldots+\frac{x^{2r}}{r}+\ldots\right\}=\Sigma\frac{x^{2n+1}}{2n+1}+3\Sigma\frac{x^{2n}}{2n}.$$

17. $\frac{e^{nx}-1}{1-e^{-x}}=e^x\frac{e^{nx}-1}{e^x-1}=e^{nx}+e^{(n-1)x}+\ldots+e^x$, and the coefficient of x^r in the last expression is obviously

$$\frac{1}{\lfloor r}\{n^r+(n-1)^r+\ldots+1^r\}.$$

Also $\frac{e^{nx}-1}{1-e^{-x}}=\dfrac{nx+\frac{n^2x^2}{\lfloor 2}+\frac{n^3x^3}{\lfloor 3}+\ldots}{x-\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}-\ldots}$

$$=\left\{n+\frac{n^2x}{\lfloor 2}+\frac{n^3x^2}{\lfloor 3}+\ldots\right\}\left[1+\left(\frac{x}{\lfloor 2}-\frac{x^2}{\lfloor 3}+\frac{x^3}{\lfloor 4}-\ldots\right)+\left\{\frac{x}{\lfloor 2}-\frac{x^2}{\lfloor 3}+\frac{x^3}{\lfloor 4}-\ldots\right\}^2+\ldots\right]$$

whence the required results.

18. $e^{e^x}=1+e^x+\frac{e^{2x}}{\lfloor 2}+\ldots+\frac{e^{kx}}{\lfloor k}+\ldots$

Hence the coefficient of x^r in e^{e^x} is $\frac{1}{\lfloor r}\Sigma\frac{m^r}{\lfloor m}$.

Also $e^{e^x}=e^{1+x+\frac{x^2}{\lfloor 2}+\ldots}=e\left[1+\left(x+\frac{x^2}{\lfloor 2}+\ldots\right)\right.$

$$\left.+\frac{1}{\lfloor 2}\left(x+\frac{x^2}{\lfloor 2}+\ldots\right)^2+\frac{1}{\lfloor 3}\left(x+\frac{x^2}{\lfloor 2}+\ldots\right)^3+\ldots\right].$$

Hence $\frac{1}{\lfloor 3}\Sigma\frac{m^3}{\lfloor m}=$ coefficient of x^3 in the last expression

$$=e\left\{\frac{1}{\lfloor 3}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}\right\}=\frac{5e}{\lfloor 3}.$$

Also $\frac{1}{\lfloor 4}\Sigma\frac{m^4}{\lfloor m}=e\left\{\frac{1}{\lfloor 4}+\frac{1}{\lfloor 2}\left(\frac{2}{\lfloor 3}+\frac{1}{4}\right)+\frac{3}{\lfloor 3\lfloor 2}+\frac{1}{\lfloor 4}\right\}=\frac{15e}{\lfloor 4}.$

19. $1+\frac{n}{1^2}+\frac{n(n-1)}{1^2 2^2}+\ldots$ is the coefficient of x^n in

$$\left\{x^n+nx^{n-1}+\frac{n(n-1)}{1.2}x^{n-2}+\ldots\right\}\times\left\{1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\right\},$$

that is in $(1+x)^n e^x=\frac{1}{e}(1+x)^n e^{1+x}$

$$=\frac{1}{e}\left[(1+x)^n+(1+x)^{n+1}+\frac{(1+x)^{n+2}}{\lfloor 2}+\frac{(1+x)^{n+3}}{\lfloor 3}+\ldots\right],$$

in which the coefficient of x^n is

$$\frac{1}{e}\left[1+\frac{n+1}{\lfloor 1\lfloor 1}+\frac{(n+2)(n+1)}{\lfloor 2\lfloor 2}+\frac{(n+3)(n+2)(n+1)}{\lfloor 3\lfloor 3}+\ldots\right].$$

20. $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n}=2\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right\}$

$$+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}.$$

Hence $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n}.$

But, when n is very great, the series on the right is equal to $\log_e 2$.

21. We know that $e^{\frac{1}{1}}>1+1$. Suppose then that $e^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}>1+n$; then $e^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{n+1}}>(1+n)\,e^{\frac{1}{1+n}}>(1+n)\left\{1+\frac{1}{n+1}+\ldots\right\}>1+n+1.$

Hence the theorem will be true for $n+1$, if it be true for n. But we know that the theorem is true when $n=1$; it is therefore true when $n=2$; and so on indefinitely.

Since $e^{\frac{1}{1}+\ldots+\frac{1}{n}}>1+n$, it follows that $\log_e(1+n)<\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}$.

Next to prove that $1+\log_e(1+n)>\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}$,

or $e(1+n) > e^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}$, or $1+n > e^{\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}$.

Since $e^{\frac{1}{r}} = 1+\frac{1}{r}+\frac{1}{\underline{|2}}\frac{1}{r^2}+\ldots < 1+\frac{1}{r}+\frac{1}{r^2}+\ldots < \frac{r}{r-1}$;

$$\therefore\ e^{\frac{1}{2}} \cdot e^{\frac{1}{3}} \ldots e^{\frac{1}{n}} < \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1} < n < n+1.$$

22. Put $-x=a,\ -y=b$ and $x+y=c$;

then $a+b+c=0,\ xy(x+y)=abc$

and $x^2+xy+y^2 = -(bc+ca+ab)$.

Since $a+b+c=0$, we have as in Art. 308, Ex. 3,

$$\frac{1}{r}(a^r+b^r+c^r) = \text{coefficient of } x^r \text{ in}$$

$$(px^2+qx^3)+\frac{1}{2}(px^2+qx^3)^2+\frac{1}{3}(px^2+qx^3)^3+\ldots\ldots$$

where $p = -(bc+ca+ab)$ and $q=abc$.

Hence (i) $\frac{1}{7}(a^7+b^7+c^7) = \frac{1}{3}\cdot 3p^2q = abc\,(bc+ca+ab)^2$.

(ii) $\frac{1}{11}(a^{11}+b^{11}+c^{11}) = p^4q+pq^3 = pq(p^3+q^2)$

$$= abc\,(bc+ca+ab)\{(bc+ca+ab)^3+a^2b^2c^2\}.$$

(iii) $\frac{1}{13}(a^{13}+b^{13}+c^{13}) = 2p^2q^3+p^5q = p^2q\,(2q^2+p^4)$

$$= abc\,(bc+ca+ab)^2\{(bc+ca+ab)^4+2a^2b^2c^2\}.$$

23. Put $-x=a,\ -y=b$ and $x+y=c$;

then $a+b+c=0,\ xy(x+y)=abc$,

and $x^2+xy+y^2 = -(bc+ca+ab)$.

Hence, as in Art. 308, Ex. 3, if $p=-(bc+ca+ab)$ and $q=abc$,

$$\frac{1}{2n}(a^{2n}+b^{2n}+c^{2n}) = \text{coefficient of } x^{2n} \text{ in } \Sigma\frac{1}{r}(px^2+qx^3)^r,$$

that is in

$$\frac{1}{n}x^{2n}(p+qx)^n+\frac{1}{n-1}x^{2n-2}(p+qx)^{n-1}+\frac{1}{n-2}x^{2n-4}(p+qx)^{n-2}+\ldots\ldots$$

$$\frac{1}{n-r}x^{2n-2r}(p+qx)^{n-r}+\ldots.$$

Hence $\frac{1}{2n}(a^{2n}+b^{2n}+c^{2n})$

$$=\frac{1}{n}p^n+\frac{n-2}{\lfloor 2}p^{n-2}q^2+\frac{(n-3)(n-4)(n-5)}{\lfloor 4}p^{n-6}q^4+\ldots\ldots$$
$$+\frac{(n-r-1)\ldots\ldots(n-3r+1)}{\lfloor 2r}p^{n-3r}q^{2r}+\ldots\ldots$$

24. Put $b-c=x$, $c-a=y$ and $a-b=z$; then $x+y+z=0$, and it is easy to prove that

$$(b-c)^3+(c-a)^3+(a-b)^3=3xyz,\ (b-c)^2+(c-a)^2+(a-b)^2=-2(yz+zx+xy),$$

and that $(b-c)^4+(c-a)^4+(a-b)^4=2(yz+zx+xy)^2$.

Now it is proved in Art. 308, Ex. 3, that if $x+y+z=0$, and numerical factors are neglected, $x^n+y^n+z^n$ is divisible by $xyz\,(yz+zx+xy)$ when n is of the form $6m-1$ and by $xyz\,(yz+zx+xy)^2$ when n is of the form $6m+1$. This proves (ii) and (iii), and it follows at once from Art. 88 that

$(b-c)^{2n+1}+(c-a)^{2n+1}+(a-b)^{2n+1}$ is divisible by $(b-c)(c-a)(a-b)$,

that is by $\frac{1}{3}\{(b-c)^3+(c-a)^3+(a-b)^3\}$.

XXXII. Pages 393—394.

1. $\log \sqrt[20]{105}=\frac{1}{20}\log 105=\frac{1}{20}(2{\cdot}0211893)={\cdot}10105946=\log 1{\cdot}2620.$

Hence $\sqrt[20]{125}=1{\cdot}262$.

2. $\log \sqrt[10]{51}=\frac{1}{10}\log 51=\frac{1}{10}(1{\cdot}7075702)={\cdot}17075702.$

Now $\log 1{\cdot}4816={\cdot}1707310$ and $\log 1{\cdot}4817={\cdot}1707603$; $\therefore$ difference for ·0001 is ·0000293, and difference for $\cdot 0000x$ is ·0000260, therefore

$$x=\frac{260}{293}\times 10=8{\cdot}8,$$

which is nearer 9 than 8. Hence the nearest approximation we can find to $\sqrt[10]{51}$ is 1·48169.

3. $A=100\left(1+\frac{1}{20}\right)^{50}$; $\therefore \log A=\log 100+50(\log 105-\log 100)$

$$=2+50(2{\cdot}0211893-2)=3{\cdot}059465.$$

Now $\log 1{\cdot}1467={\cdot}0594498$ and $\log 1{\cdot}1468={\cdot}0594877$, therefore difference for ·0001 is ·0000379 and difference for $\cdot 0000x$ is ·0000152; hence $x=\frac{1520}{379}=4$. Hence 3·059465 is the logarithm of 1146·74: thus the amount required is £1146·74.

4. At 5 per cent. per annum the amount of £A at the end of 15 years will be £$A\left(1+\frac{5}{100}\right)^{15}$, which will be more than £2A if $\left(1+\frac{5}{100}\right)^{15}>2$, or if $15\log(105-\log 100)>\log 2$, that is if $15\,(\cdot0211893)>\cdot3010300$.

At 4 per cent. per annum the amount of £A at the end of 18 years will be £$A\left(1+\frac{4}{100}\right)^{18}$, which will be greater than £2A if $\left(1+\frac{4}{100}\right)^{18}>2$, or if $18\log 1\cdot04>\log 2$, that is $18\,(\cdot0170333)>\cdot3010300$.

5. The amount is $500\left(1+\frac{2}{100}\right)^{20}$.

Hence $\log A=\log 500+20\log 1\cdot02=3-\cdot3010300+\cdot1720040=2\cdot8709740$.

Now $\log 7\cdot4297=\cdot8709713$ and $\log 7\cdot4298=\cdot8709771$, the difference for ·0001 being ·0000058; also the difference for ·0000x is ·0000027;

$$\therefore\quad x=\frac{27}{58}\times10=5 \text{ most nearly.}$$

Hence $2\cdot8709740$ is $\log 742\cdot975$, the required amount is therefore

$$£742\cdot975=£742-19s.\ 6d.$$

6. The yearly gain is 33 per 1000. Hence in 22 years the population will be $P\left(1+\frac{33}{1000}\right)^{22}$, which will be greater than $2P$ if $\left(1+\frac{33}{1000}\right)^{22}>2$, or if $22\log 1\cdot033>\log 2$, that is if $22\times\cdot0141003>\cdot3010300$, which is easily seen to be the case.

7. The amount of the first £30 will be $30\,(1+\cdot025)^{20}$, the amount of the second £30 will be $30\,(1+\cdot025)^{19}$, and so for the rest. Hence the total amount will be

$$30\{1\cdot025^{20}+1\cdot025^{19}+\ldots+1\cdot025^{2}+1\cdot025\}$$
$$=30\,(1\cdot025)\left\{\frac{1\cdot025^{20}-1}{1\cdot025-1}\right\}=1230\,\{1\cdot025^{20}-1\}.$$

Now $\log 1\cdot025^{20}=20\log 1\cdot025=20\times\cdot0107239=\cdot2144780$. Hence

$$1\cdot025^{20}=1\cdot6386x,\text{ where } x=\frac{50}{265}\times10=2 \text{ nearly.}$$

Hence the amount required $=£1230\times\cdot63862=£785\cdot5026=£785-10s.$ nearly.

8. From Art. 314, III., the required present value

$$=\frac{100}{\frac{1}{25}}\left\{1-\frac{1}{\left(1+\frac{1}{25}\right)^{40}}\right\}=2500\,\{1-(1\cdot04)^{-40}\}.$$

Now $\log(1\cdot04)^{-40}=-40\log(1\cdot04)=-\cdot681332=\bar{1}\cdot318668=\log\cdot20829$.

Hence the present value $=£2500\,(1-\cdot20829)=£1979\cdot275=£1979-5s.\ 6d.$

9. If A be the required yearly payment, we have from Art. 310, III.,

$$30000 = \frac{A}{\frac{1}{25}}\left\{1 - \frac{1}{\left(1+\frac{1}{25}\right)^{30}}\right\} = 25A\{1-(1{\cdot}04)^{-30}\}.$$

Now $\log 1{\cdot}04^{-30} = -30\log 1{\cdot}04 = -{\cdot}510999 = \bar{1}{\cdot}489001 = \log {\cdot}30832.$

Hence $30000 = 25A(1-{\cdot}30832)$;

$$\therefore\ A = £\frac{1200}{{\cdot}69168} = £1734{\cdot}9.$$

10. The fine is the present value of £(70 − 10) a year for 14 years beginning at the end of 26 years. Hence

$$F = 60\{(1{\cdot}06)^{-27} + (1{\cdot}06)^{-28} + \text{... to 14 terms}\}$$

$$= 60(1{\cdot}06)^{-27}\left\{\frac{1-(1{\cdot}06)^{-14}}{1-(1{\cdot}06)^{-1}}\right\} = 1000(1{\cdot}06)^{-26}\{1-(1{\cdot}06)^{-14}\}.$$

Hence $\log F = 3 - 26\log 1{\cdot}06 + \log\{1-(1{\cdot}06)^{-14}\}$.

Now $\log(1{\cdot}06)^{-14} = -14\log 1{\cdot}06 = -{\cdot}3542826 = \bar{1}{\cdot}6457174 = \log {\cdot}4423.$

Hence $\log\{1-(1{\cdot}06)^{-14}\} = \log(1-{\cdot}4423) = \log {\cdot}5577 = \bar{1}{\cdot}7464006.$

Hence $\log F = 3 + \bar{1}{\cdot}3420466 + \bar{1}{\cdot}7464006 = 2{\cdot}0884472.$

Hence $F = £122{\cdot}58$ nearly.

XXXIII. Pages 407—409.

Here $u_n = (3n+1)(3n+4)(3n+7)$,

$v_n = (3n+1)(3n+4)(3n+7)(3n+10)$, $v_0 = 1\ .\ 4\ .\ 7\ .\ 10$, $r=3$, $b=3$.

Hence $S_n = \frac{1}{12}\{(3n+1)(3n+4)(3n+7)(3n+10) - 1\ .\ 4\ .\ 7\ .\ 10\}$.

2. Here $u_n = \frac{1}{(4n-1)(4n+3)(4n+7)}$, $v_n = \frac{1}{(4n+3)(4n+7)}$,

$$v_0 = \frac{1}{3\ .\ 7},\ r=3,\ b=4.$$

Hence $S_n = \frac{1}{8}\left\{\frac{1}{3\ .\ 7} - \frac{1}{(4n+3)(4n+7)}\right\}$, and $S_\infty = \frac{1}{168}$.

3. Here $u_n = n(n+2)(n+3) = n(n+1)(n+2) + 2n(n+1) + 2n.$

Hence $S_n = \frac{1}{4} n(n+1)(n+2)(n+3) + \frac{2}{3} n(n+1)(n+2) + n(n+1).$

4. Here $u_n = (2n-1)(2n+3) = 4n(n+1) - 3.$

Hence $S_n = \frac{4}{3} n(n+1)(n+2) - 3n.$

5. Here $u_n = n(n+1)(2n+1) = n(n+1)(n+2) + (n-1)n(n+1).$

Hence $S_n = \frac{1}{4} n(n+1)(n+2)(n+3) + \frac{1}{4}(n-1)n(n+1)(n+2).$

6. Here $u_n = n(n+1)^2 = n(n+1)(n+2) - n(n+1).$

Hence $S_n = \frac{1}{4} n(n+1)(n+2)(n+3) - \frac{1}{3} n(n+1)(n+2).$

7. Here

$$u_n = (2n-1)(2n+1)^2 = (2n-1)(2n+1)(2n+3) - 2(2n-1)(2n+1).$$

Hence

$$S_n = \frac{1}{8}\{(2n-1)(2n+1)(2n+3)(2n+5) + 1 \cdot 3 \cdot 5\} - \frac{2}{6}\{(2n-1)(2n+1)(2n+3) + 1 \cdot 3\}.$$

8. Here $u_n = \dfrac{1}{(2n-1)(2n+1)(2n+5)} = \dfrac{2n+3}{(2n-1)(2n+1)(2n+3)(2n+5)}$

$$= \frac{1}{(2n-1)(2n+1)(2n+3)} - \frac{2}{(2n-1)(2n+1)(2n+3)(2n+5)}.$$

Hence $S_n = \frac{1}{4}\left\{\dfrac{1}{1 \cdot 3} - \dfrac{1}{(2n+1)(2n+3)}\right\}$

$$- \frac{2}{6}\left\{\frac{1}{1 \cdot 3 \cdot 5} - \frac{1}{(2n+1)(2n+3)(2n+5)}\right\},$$

and $$S_\infty = \frac{1}{12} - \frac{1}{45} = \frac{11}{180}.$$

9. Here $u_n = \dfrac{1}{n(n+2)(n+3)} = \dfrac{n+1}{n(n+1)(n+2)(n+3)}$

$$= \frac{1}{(n+1)(n+2)(n+3)} + \frac{1}{n(n+1)(n+2)(n+3)}.$$

Hence $S_n=\frac{1}{2}\left\{\frac{1}{2\,.\,3}-\frac{1}{(n+2)(n+3)}\right\}+\frac{1}{3}\left\{\frac{1}{1\,.\,2\,.\,3}-\frac{1}{(n+1)(n+2)(n+3)}\right\}$,

and $$S_\infty=\frac{1}{12}+\frac{1}{18}=\frac{5}{36}.$$

10. Here $u_n=\frac{n+3}{n(n+1)(n+2)}=\frac{1}{n(n+1)}+\frac{1}{n(n+1)(n+2)}$.

Hence $S_n=\left\{1-\frac{1}{n+1}\right\}+\frac{1}{2}\left\{\frac{1}{1\,.\,2}-\frac{1}{(n+1)(n+2)}\right\}$,

and $$S_\infty=1+\frac{1}{4}.$$

11. Here $u_n=\frac{n}{(2n-1)(2n+1)(2n+3)}$

$$=\frac{1}{2(2n+1)(2n+3)}+\frac{1}{2(2n-1)(2n+1)(2n+3)}.$$

Hence $S_n=\frac{1}{4}\left\{\frac{1}{3}-\frac{1}{2n+3}\right\}+\frac{1}{8}\left\{\frac{1}{1\,.\,3}-\frac{1}{(2n+1)(2n+3)}\right\}$,

and $$S_\infty=\frac{1}{12}+\frac{1}{24}.$$

12. Here

$$u_n=\frac{n+2}{n(n+1)(n+3)}=\frac{(n+2)^2}{n(n+1)(n+2)(n+3)}=\frac{n(n+3)+(n+3)+1}{n(n+1)(n+2)(n+3)}$$

$$=\frac{1}{(n+1)(n+2)}+\frac{1}{n(n+1)(n+2)}+\frac{1}{n(n+1)(n+2)(n+3)}.$$

Hence $S_n=\left\{\frac{1}{2}-\frac{1}{n+2}\right\}+\frac{1}{2}\left\{\frac{1}{1\,.\,2}-\frac{1}{(n+1)(n+2)}\right\}$

$$+\frac{1}{3}\left\{\frac{1}{1\,.\,2\,.\,3}-\frac{1}{(n+1)(n+2)(n+3)}\right\},$$

and $$S_\infty=\frac{1}{2}+\frac{1}{4}+\frac{1}{18}.$$

13. Here $$u_n=\frac{1}{1+2+3+\ldots+n}=\frac{2}{n(n+1)};$$

$$\therefore\ S_n=2\left(\frac{1}{1}-\frac{1}{n+1}\right)\text{ and }S_\infty=2.$$

14. Here $u_n=\frac{\frac{1}{6}n(n+1)(2n+1)}{n}=\frac{1}{6}(n+1)(2n+1)$

$$=\frac{1}{3}n(n+1)+\frac{1}{6}(n+1).$$

Hence $S_n=\frac{1}{9}n(n+1)(n+2)+\frac{1}{12}\{(n+1)(n+2)-1\cdot 2\}$

$$=\frac{1}{36}(n+1)(n+2)(4n+3)-\frac{1}{6}.$$

15. Here

$$u_n=\frac{1}{6}n^2(n+1)(2n+1)=\frac{1}{6}n(n+1)\{2(n+2)(n+3)-9(n+2)+6\}$$

$$=\frac{1}{3}n(n+1)(n+2)(n+3)-\frac{3}{2}n(n+1)(n+2)+n(n+1).$$

Hence

$$S_n=\frac{1}{15}n(n+1)(n+2)(n+3)(n+4)-\frac{3}{8}n(n+1)(n+2)(n+3)$$

$$+\frac{1}{3}n(n+1)(n+2).$$

16. Here $u_n=\{a+(n-1)b\}^2=a^2+2ab(n-1)+b^2(n-1)^2$.

Hence $S_n=na^2+ab(n-1)n+\frac{1}{6}b^2(n-1)n(2n-1)$.

17. $u_n=\{a+(n-1)b\}^3=a^3+3(n-1)a^2b+3(n-1)^2ab^2+(n-1)^3b^3$.

Hence $S_n=na^3+\frac{3}{2}(n-1)na^2b+\frac{1}{2}(n-1)n(2n-1)ab^2+\frac{1}{4}(n-1)^2n^2b^3$.

18. $u_n=(2n-1)^2=4n(n-1)+1$.

Hence $u_n=\frac{4}{3}(n-1)n(n+1)+n$.

19. $u_n=(4n-3)^2=16n^2-24n+9=16n(n-1)-8n+9$.

Hence $S_n=\frac{16}{3}(n-1)n(n+1)-4n(n+1)+9n$.

20. $1^2-2^2+3^2-4^2+\ldots+(2n+1)^2=1+(3+2)+(5+4)+\ldots(2n+1+2n)$

$$=1+5+9+13+\ldots+(4n+1)=(n+1)(2n+1).$$

21. $1^2-2^2+3^2-4^2+\ldots-(2n)^2=-(1+2)-(3+4)-\ldots-(2n+2n-1)$

$$=-\{3+7+11+\ldots+(4n-1)\}=-n(2n+1).$$

22. $1^3-2^3+3^3-\ldots+(2n+1)^3$

$$=1^3+2^3+3^3+\ldots+(2n+1)^3-2\{2^3+4^3+6^3+\ldots+(2n)^3\}$$

$$=\frac{1}{4}(2n+1)^2(2n+2)^2-\frac{16}{4}n^2(n+1)^2=(n+1)^2(4n+1).$$

23. $S = 1 \cdot n + 2(n-1) + 3(n-2) + \ldots\ldots + n\{n-(n-1)\}$

$= n(1+2+\ldots+n) - \{1 \cdot 2 + 2 \cdot 3 + \ldots + (n-1)n\}$

$= \frac{1}{2}n^2(n+1) - \frac{1}{3}(n-1)n(n+1) = \frac{1}{6}n(n+1)(n+2).$

24. $S = n^2 + (n^2-1^2) + (n^2-2^2) + \ldots + \{n^2-(n-2)^2\} + \{n^2-(n-1)^2\}$

$= n \cdot n^2 - \frac{1}{6}(n-1)n(2n-1).$

25. $S_n = ab + \{ab-(a+b)+1^2\} + \{ab-2(a+b)+2^2\} + \ldots$
$+ \{ab-(n-1)(a+b)+(n-1)^2\}$

$= nab - \frac{1}{2}(n-1)n(a+b) + \frac{1}{6}(n-1)n(2n-1).$

26. (i) The theorem is clearly true when n is 1; and assuming its truth for n it will be true for $n+1$ provided

$$5(n+1)^4 = 6 \cdot \frac{1}{2}(n+1)(n+2) \cdot \frac{1}{6}(n+1)(n+2)(2n+3)$$
$$- 6 \cdot \frac{1}{2}n(n+1) \cdot \frac{1}{6}n(n+1)(2n+1) - (n+1)^2,$$

which is easily seen to be the case. Hence, by induction, the theorem is true for all values of n.

(ii) Assuming that the theorem is true for any value of n, it will be true for the next greater value, provided

$$(n+1)^7 + (n+1)^5 = 2\{S_n^3 + (n+1)^3\}^2 - 2(S_n^3)^2$$
$$= 4(n+1)^3 \cdot S_n^3 + 2(n+1)^6$$
$$= (n+1)^3 \cdot n^2(n+1)^2 + 2(n+1)^6,$$

which is obvious.

Hence, as the theorem is obviously true when $n=1$, it is by induction true for all values of n.

27. (i) $$u_n = \frac{n}{(n+1)(n+2)} 2^n = \frac{2^{n+1}}{n+2} - \frac{2^n}{n+1}.$$

Hence

$$S_n = \left(\frac{2^2}{3} - \frac{2}{2}\right) + \left(\frac{2^3}{4} - \frac{2^2}{3}\right) + \ldots\ldots + \left(\frac{2^{n+1}}{n+2} - \frac{2^n}{n+1}\right) = \frac{2^{n+1}}{n+2} - 1.$$

(ii) $$u_n = \frac{n+2}{n(n+1)} \frac{1}{2^n} = \left(\frac{2}{n} - \frac{1}{n+1}\right) \frac{1}{2^n} = \frac{1}{n2^{n-1}} - \frac{1}{(n+1)2^n}.$$

Hence $S_n=\left(\frac{1}{1}-\frac{1}{2\,.\,2^1}\right)+\left(\frac{1}{2\,.\,2^1}-\frac{1}{3\,.\,2^2}\right)+\left(\frac{1}{3\,.\,2^2}-\frac{1}{4\,.\,2^3}\right)+\ldots$

$$+\left\{\frac{1}{n\,.\,2^{n-1}}-\frac{1}{(n+1)\,2^n}\right\}=1-\frac{1}{(n+1)\,2^n}.$$

(iii) $u_n=\frac{n+3}{n(n+1)}\left(\frac{2}{3}\right)^n=\frac{3(n+1)-2n}{n(n+1)}\left(\frac{2}{3}\right)^n=\frac{2}{n}\left(\frac{2}{3}\right)^{n-1}-\frac{2}{n+1}\left(\frac{2}{3}\right)^n.$

Hence $S_n=\left\{\frac{2}{1}-\frac{2}{2}\left(\frac{2}{3}\right)^1\right\}+\left\{\frac{2}{2}\left(\frac{2}{3}\right)^1-\frac{2}{3}\left(\frac{2}{3}\right)^2\right\}+\ldots$

$$+\left\{\frac{2}{n}\left(\frac{2}{3}\right)^{n-1}-\frac{2}{n+1}\left(\frac{2}{3}\right)^n\right\}=2-\frac{2}{n+1}\left(\frac{2}{3}\right)^n.$$

(iv) $u_n=\frac{n+7}{n(n+1)(n+2)}\left(\frac{5}{7}\right)^n=\frac{7(n+2)-5n}{2n(n+1)(n+2)}\left(\frac{5}{7}\right)^n$

$$=\frac{5}{2n(n+1)}\left(\frac{5}{7}\right)^{n-1}-\frac{5}{2(n+1)(n+2)}\left(\frac{5}{7}\right)^n.$$

Hence $2S_n=\left\{\frac{5}{1\,.\,2}-\frac{5}{2\,.\,3}\cdot\frac{5}{7}\right\}+\left\{\frac{5}{2\,.\,3}\left(\frac{5}{7}\right)-\frac{5}{3\,.\,4}\left(\frac{5}{7}\right)^2\right\}+\ldots$

$$+\left\{\frac{5}{n(n+1)}\left(\frac{5}{7}\right)^{n-1}-\frac{5}{(n+1)(n+2)}\left(\frac{5}{7}\right)^n\right\}=\frac{5}{1\,.\,2}-\frac{5}{(n+1)(n+2)}\left(\frac{5}{7}\right)^n.$$

(v) $u_n=\frac{n+8}{n(n+1)(n+2)}\left(\frac{3}{4}\right)^n=\frac{4(n+2)-3n}{n(n+1)(n+2)}\left(\frac{3}{4}\right)^n$

$$=\frac{3}{n(n+1)}\left(\frac{3}{4}\right)^{n-1}-\frac{3}{(n+1)(n+2)}\left(\frac{3}{4}\right)^n.$$

Hence $S_n=\left\{\frac{3}{1\,.\,2}-\frac{3}{2\,.\,3}\left(\frac{3}{4}\right)\right\}+\left\{\frac{3}{2\,.\,3}\left(\frac{3}{4}\right)-\frac{3}{3\,.\,4}\left(\frac{3}{4}\right)^2\right\}+\ldots$

$$+\left\{\frac{3}{n(n+1)}\left(\frac{3}{4}\right)^{n-1}-\frac{3}{(n+1)(n+2)}\left(\frac{3}{4}\right)^n\right\}=\frac{3}{2}-\frac{3}{(n+1)(n+2)}\left(\frac{3}{4}\right)^n.$$

(vi) $u_n=\frac{n+14}{n(n+1)(n+2)}\left(\frac{6}{7}\right)^n=\frac{7(n+2)-6n}{n(n+1)(n+2)}\left(\frac{6}{7}\right)^n$

$$=\frac{6}{n(n+1)}\left(\frac{6}{7}\right)^{n-1}-\frac{6}{(n+1)(n+2)}\left(\frac{6}{7}\right)^n.$$

Hence $S_n = \left\{\frac{6}{1\,.\,2} - \frac{6}{2\,.\,3}\cdot\frac{6}{7}\right\} + \left\{\frac{6}{2\,.\,3}\cdot\frac{6}{7} - \frac{6}{3\,.\,4}\left(\frac{6}{7}\right)^2\right\} + \ldots$

$$+ \left\{\frac{6}{n\,(n+1)}\left(\frac{6}{7}\right)^{n-1} - \frac{6}{(n+1)\,(n+2)}\left(\frac{6}{7}\right)^n\right\} = 3 - \frac{6}{(n+1)\,(n+2)}\left(\frac{6}{7}\right)^n.$$

28 and 29. $(\Sigma a)^2 = \Sigma a^2 + 2\Sigma ab$;

also $(\Sigma a)^3 = \Sigma a^3 + 3\Sigma a^2 b + 6\Sigma abc$

$$= 3\Sigma a^2\,.\,\Sigma a - 2\Sigma a^3 + 6\Sigma abc.$$

If $a=1,\ b=2,\ c=3, \ldots k=n$.

$$\Sigma a = \frac{1}{2}n\,(n+1), \quad \Sigma a^2 = \frac{1}{6}n\,(n+1)\,(2n+1), \quad \text{and} \quad \Sigma a^3 = \frac{1}{4}n^2\,(n+1)^2.$$

Hence sum of products two together

$$= \frac{1}{2}\left\{\frac{1}{4}n^2\,(n+1)^2 - \frac{1}{6}n\,(n+1)\,(2n+1)\right\} = \frac{1}{24}(n-1)\,n\,(n+1)\,(3n+2).$$

Also sum of the products three together

$$= \frac{1}{6}\left\{\frac{1}{8}n^3\,(n+1)^3 - 3\,.\,\frac{1}{6}n\,(n+1)\,(2n+1)\,.\,\frac{1}{2}n\,(n+1) + 2\,.\,\frac{1}{4}n^2\,(n+1)^2\right\}$$

$$= \frac{1}{48}n^2\,(n+1)^2\,\{n\,(n+1) - 2\,(2n+1) + 4\} = \frac{1}{48}n^2\,(n+1)^2\,(n-1)\,(n-2).$$

30. The sum required

$$= \frac{1}{2}\left\{(1^2 + 2^2 + \ldots + n^2)^2 - (1^4 + 2^4 + \ldots + n^4)\right\}$$

$$= \frac{1}{2}\left[\frac{1}{36}n^2\,(n+1)^2\,(2n+1)^2 - \frac{1}{5}\left\{3n\,(n+1)\,.\,\frac{1}{6}n\,(n+1)\,(2n+1) - \frac{1}{6}n\,(n+1)\,(2n+1)\right\}\right] \text{ from 26.}$$

$$= \frac{1}{360}n\,(n+1)\,(2n+1)\,[5n\,(n+1)\,(2n+1) - 18\,n\,(n+1) + 6]$$

$$= \frac{1}{360}n\,(n+1)\,(2n+1)\,(n-1)\,(2n-1)\,(5n+6).$$

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1. (i)
$$\frac{4}{5}=\frac{1}{2}\left\{\frac{4.7}{5}-\frac{4}{1}\right\},$$
$$\frac{4.7}{5.8}=\frac{1}{2}\left\{\frac{4.7.10}{5.8}-\frac{4.7}{5}\right\},$$
$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$
$$\frac{4.7\ldots(3n+1)}{5.8\ldots(3n+2)}=\frac{1}{2}\left\{\frac{4.7\ldots(3n+4)}{5.8\ldots(3n+2)}-\frac{4.7\ldots(3n+1)}{5.8\ldots(3n-1)}\right\};$$
$$\therefore S_n=\frac{1}{2}\left\{\frac{4.7\ldots(3n+4)}{5.8\ldots(3n+2)}-\frac{4}{1}\right\}$$
$$=\frac{4.7\ldots(3n+4)}{2.5\ldots(3n+2)}-2.$$

(ii)
$$\frac{2}{4}=\frac{2.5}{4}-\frac{2}{1},$$
$$\frac{2.5}{4.7}=\frac{2.5.8}{4.7}-\frac{2.5}{4},$$
$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$
$$\frac{2.5\ldots(3n-1)}{4.7\ldots(3n+1)}=\frac{2.5\ldots(3n+2)}{4.7\ldots(3n+1)}-\frac{2.5\ldots(3n-1)}{4.7\ldots(3n-2)};$$
$$\therefore S_n=\frac{2.5\ldots(3n+2)}{4.7\ldots(3n+1)}-2.$$

(iii)
$$\frac{3}{8}=\frac{1}{3}\left\{\frac{3}{1}-\frac{3.5}{8}\right\},$$
$$\frac{3.5}{8.10}=\frac{1}{3}\left\{\frac{3.5}{8}-\frac{3.5.7}{8.10}\right\},$$
$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$
$$\frac{3.5\ldots(2n+1)}{8.10\ldots(2n+6)}=\frac{1}{3}\left\{\frac{3.5\ldots(2n+1)}{8.10\ldots(2n+4)}-\frac{3.5\ldots(2n+3)}{8.10\ldots(2n+6)}\right\};$$
$$\therefore S_n=\frac{1}{3}\left\{\frac{3}{1}-\frac{3.5\ldots(2n+3)}{8.10\ldots(2n+6)}\right\}$$
$$=1-\frac{5.7\ldots(2n+3)}{8.10\ldots(2n+6)}.$$

When n is increased without limit, we have $S_\infty=1$.

(iv) $$\frac{11}{14}=\frac{11}{1}-\frac{11.13}{14},$$

$$\frac{11.13}{14.16}=\frac{11.13}{14}-\frac{11.13.15}{14.16},$$

$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$

$$\frac{11.13\ldots(2n+9)}{14.16\ldots(2n+12)}=\frac{11.13\ldots(2n+9)}{14.16\ldots(2n+10)}-\frac{11.13\ldots(2n+11)}{14.16\ldots(2n+12)};$$

$$\therefore\ S_n=11-\frac{11.13\ldots(2n+11)}{14.16\ldots(2n+12)}.$$

Also, increasing n without limit, we have $S_\infty=11$.

2. Let u_n be the nth term of the series.

Let v_n „ „ „ 1st order of differences.

Let w_n „ „ „ 2nd „ „

(i) The first order of differences is 0, 6, 12, 18,...

$$\therefore\ v_n=6(n-1);$$

$$\therefore\ u_n-u_{n-1}=6(n-2),\ u_{n-1}-u_{n-2}=6(n-3),$$

$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$

$$u_3-u_2=6.1;\ \text{also } u_2=2.$$

$$\therefore\ u_n=2+6\{1+2+3+\ldots+(n-2)\}$$

$$=2+3(n-1)(n-2);$$

$$\therefore\ S_n=2n+n(n-1)(n-2).$$

(ii) The first order of differences is 7, 5, 3, 1, −1,...

$$\therefore\ v_n=7-2(n-1).$$

$$\therefore\ u_n-u_{n-1}=7-2(n-2),\ u_{n-1}-u_{n-2}=7-2(n-3),$$

$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$

$$u_3-u_2=7-2.1.$$

Also $$u_2=14;$$

$$\therefore\ u_n=14+7(n-2)-(n-1)(n-2)$$

$$=7n-(n-1)(n-2);$$

$$\therefore\ S_n=\frac{7}{2}n(n+1)-\frac{1}{3}n(n-1)(n-2).$$

(iii) The first order of differences is 3, 7, 15, 31, 63,...

The second order is 4, 8, 16, 32,...

$$\therefore w_n = 2^{n+1};$$

$$\therefore v_n - v_{n-1} = 2^n,\ v_{n-1} - v_{n-2} = 2^{n-1},$$

$$\ldots\ldots = \ldots\ldots,$$

$$v_2 - v_1 = 2^2. \quad \text{Also } v_1 = 3 = 1 + 2.$$

Hence $$v_n = 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1;$$

$$\therefore u_n - u_{n-1} = 2^n - 1,\ u_{n-1} - u_{n-2} = 2^{n-1} - 1,$$

$$\ldots\ldots = \ldots\ldots,$$

$$u_2 - u_1 = 2^2 - 1;\ \text{also } u_1 = 1 = 2 - 1.$$

Hence $$u_n = (2 + 2^2 + \ldots + 2^n) - n = 2^{n+1} - 2 - n.$$

$$S_n = (2^2 + 2^3 + \ldots + 2^{n+1}) - \{3 + 4 + 5 + \ldots + (n+2)\}$$

$$= 2^{n+2} - 4 - \left\{\frac{1}{2}(n+2)(n+3) - 3\right\} = 2^{n+2} - 1 - \frac{1}{2}(n+2)(n+3).$$

(iv) The first order of differences is $-1, 1, 7, 21, 51, 113,\ldots$

The second order is 2, 6, 14, 30, 62,...

The third order is 4, 8, 16, 32,..., which is a geometrical progression.

$$\therefore w_n - w_{n-1} = 2^n,\ w_{n-1} - w_{n-2} = 2^{n-1},$$

$$\ldots\ldots = \ldots\ldots,$$

$$w_2 - w_1 = 2^2;\ \text{also } w_1 = 2.$$

Hence $$w_n = 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 2;$$

$$\therefore v_n - v_{n-1} = 2^n - 2,\ v_{n-1} - v_{n-2} = 2^{n-1} - 2;$$

$$\ldots\ldots = \ldots\ldots,$$

$$v_2 - v_1 = 2^2 - 2;\ \text{also } v_1 = -1.$$

Hence $$v_n = (2^2 + 2^3 + \ldots + 2^n) - 2(n-1) - 1 = 2^{n+1} - 4 - 2n + 2 - 1$$

$$= 2^{n+1} - 2(n+1) - 1;$$

$$\therefore u_n - u_{n-1} = 2^n - 2n - 1,\ u_{n-1} - u_{n-2} = 2^{n-1} - 2(n-1) - 1,$$

$$\ldots\ldots = \ldots\ldots,$$

$$u_2 - u_1 = 2^2 - 2 \cdot 2 - 1;\ \text{also } u_1 = 1 = 2 - 2 \cdot 0 - 1.$$

Hence $$u_n = (2 + 2^2 + \ldots + 2^n) - 2(2 + 3 + 4 + \ldots + n) - n$$

$$= 2^{n+1} - 2 - \{n(n+1) - 2\} - n$$

$$= 2^{n+1} - n(n+1) - n;$$

$$\therefore S_n=(2^2+2^3+\ldots+2^{n+1})-\frac{1}{3}n(n+1)(n+2)-\frac{1}{2}n(n+1)$$

$$=2^{n+2}-4-\frac{1}{3}n(n+1)(n+2)-\frac{1}{2}n(n+1).$$

(v) The first order of differences is 4, 10, 20, 35, 56,...

The second order is 6, 10, 15, 21,...

The third order is 4, 5, 6,...

$$\therefore\ w_n-w_{n-1}=n+2,\ w_{n-1}-w_{n-2}=n+1,$$

$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots,$$

$$w_2-w_1=4;\ \text{also } w_1=6=1+2+3.$$

Hence $$w_n=1+2+3+\ldots+(n+2)=\frac{1}{2}(n+2)(n+3).$$

$$\therefore\ v_n-v_{n-1}=\frac{1}{2}(n+1)(n+2),\ v_{n-1}-v_{n-2}=\frac{1}{2}n(n+1),$$

$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots,$$

$$v_2-v_1=\frac{1}{2}.3.4;\ \text{also } v_1=4=\frac{1}{2}(1.2+2.3).$$

Hence

$$v_n=\frac{1}{2}\{1.2+2.3+\ldots+(n+1)(n+2)\}=\frac{1}{6}(n+1)(n+2)(n+3).$$

$$\therefore\ u_n-u_{n-1}=\frac{1}{6}n(n+1)(n+2),\ u_{n-1}-u_{n-2}=\frac{1}{6}(n-1)n(n+1).$$

$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots,$$

$$u_2-u_1=\frac{1}{6}.2.3.4;\ \text{also } u_1=1=\frac{1}{6}.1.2.3.$$

Hence

$$u_n=\frac{1}{6}\{1.2.3+2.3.4+\ldots+n(n+1)(n+2)\}=\frac{1}{24}n(n+1)(n+2)(n+3);$$

$$\therefore\ S_n=\frac{1}{120}n(n+1)(n+2)(n+3)(n+4).$$

(vi) The first order of differences is 1, 27, 101, 247, 489, 851,...

The second order is 26, 74, 146, 242, 362,...

The third order is 48, 72, 96, 120,...

which is an Arithmetical Progression.

$$\therefore\ w_n - w_{n-1} = 24n,\ w_{n-1} - w_{n-2} = 24(n-1),$$

$$\ldots\ldots = \ldots\ldots,$$

$$w_2 - w_1 = 24 \cdot 2;\ \text{also } w_1 = 26 = 2 + 24.$$

Hence $w_n = 2 + 24\{1 + 2 + \ldots + n\} = 2 + 12n(n+1).$

$$\therefore\ v_n - v_{n-1} = 2 + 12(n-1)n,\ v_{n-1} - v_{n-2} = 2 + 12(n-2)(n-1),$$

$$\ldots\ldots = \ldots\ldots,$$

$$v_2 - v_1 = 2 + 12 \cdot 1 \cdot 2;\ \text{also } v_1 = 1.$$

Hence $v_n = 1 + 2(n-1) + 12\{1 \cdot 2 + 2 \cdot 3 + \ldots + (n-1)n\}$

$$= 2n - 1 + 4(n-1)n(n+1).$$

$$\therefore\ u_n - u_{n-1} = 2n - 3 + 4(n-2)(n-1)n,$$

$$u_{n-1} - u_{n-2} = 2n - 5 + 4(n-3)(n-2)(n-1),$$

$$\ldots\ldots = \ldots\ldots,$$

$$u_3 - u_2 = 3 + 4 \cdot 1 \cdot 2 \cdot 3;\ \text{also } u_2 = 2.$$

Hence $u_n = 2 + \{3 + 5 + \ldots + (2n-3)\} + (n-2)(n-1)n(n+1)$

$$= 2 + \frac{(n-2)2n}{2} + (n-2)(n-1)n(n+1)$$

$$= 2 + (n-1)n - n + (n-2)(n-1)n(n+1).$$

$$\therefore\ S_n = 2n + \frac{1}{3}(n-1)n(n+1) - \frac{1}{2}n(n+1) + \frac{1}{5}(n-2)(n-1)n(n+1)(n+2).$$

3. (i) Let the scale of relation be $1 + px + qx^2$.

Then $14 + 4p + 2q = 0$, and $52 + 14p + 4q = 0$;

$$\therefore\ p = -4,\ \text{and } q = 1.$$

Hence the generating function

$$= \frac{a_0 + (a_1 + pa_0)x}{1 + px + qx^2}$$

$$= \frac{2 + (4 - 4 \cdot 2)x}{1 - 4x + x^2} = \frac{2 - 4x}{1 - 4x + x^2}.$$

(ii) Here $11 + 3p + q = 0$, and $43 + 11p + 3q = 0$;

$$\therefore\ p = -5 \text{ and } q = 4.$$

Hence the generating function

$$= \frac{1 + (3 - 5)x}{1 - 5x + 4x^2} = \frac{1 - 2x}{1 - 5x + 4x^2}.$$

(iii) Here $40+6p+q=0$, and $288+40p+6q=0$;

$$\therefore\ p=-12 \text{ and } q=32.$$

Hence the generating function

$$=\frac{1+(6-12)x}{1-12x+32x^2}=\frac{1-6x}{1-12x+32x^2}.$$

(iv) Let the scale of relation be $1+px+qx^2+rx^3$.

Then $7+2p+q+r=0$, $14+7p+2q+r=0$, $35+14p+7q+2r=0$;

$$\therefore\ p=-\frac{14}{15},\ q=-\frac{7}{3} \text{ and } r=-\frac{14}{5}.$$

Hence the generating function

$$=\frac{a_0+(a_1+pa_0)x+(a_2+pa_1+qa_0)x^2}{1+px+qx^2+rx^3}$$

$$=\frac{1+\left(1-\frac{14}{15}\right)x+\left(2-\frac{14}{15}-\frac{7}{3}\right)x^2}{1-\frac{14}{15}x-\frac{7}{3}x^2-\frac{14}{5}x^3}$$

$$=\frac{15+x-19x^2}{15-14x-35x^2-42x^3}.$$

(v) Here

$$4^2+3^2p+2^2q+1^2r=0,\ 5^2+4^2p+3^2q+2^2r=0,\ 6^2+5^2p+4^2q+3^2r=0$$

$$\therefore\ p=-3,\ q=3 \text{ and } r=-1.$$

Hence the generating function

$$=\frac{1^2+(2^2-3\,.\,1^2)x+(3^2-3\,.\,2^2+3\,.\,1^2)x^2}{1-3x+3x^2-x^3}=\frac{1+x}{(1-x)^3}.$$

4. (i) Here $14+6p+2q=0$, $30+14p+6q=0$;

$$\therefore\ p=-3 \text{ and } q=2.$$

Hence the generating function

$$=\frac{2+(6-3\,.\,2)x}{1-3x+2x^2}=\frac{2}{(1-x)(1-2x)}=\frac{4}{1-2x}-\frac{2}{1-x}.$$

Hence the nth term $=2^{n+1}-2$,

and $S_n=2^{n+2}-4-2n$.

(ii) Here $29-5p+2q=0$, and $-89+29p-5q=0$;

$$\therefore\ p=1 \text{ and } q=-12.$$

Hence the generating function

$$=\frac{2+(-5+2)x}{1+x-12x^2}=\frac{2-3x}{(1+4x)(1-3x)}=\frac{\frac{11}{7}}{1+4x}+\frac{\frac{3}{7}}{1-3x}.$$

Hence the nth term $=\frac{11}{7}(-4)^{n-1}+\frac{3}{7}\cdot 3^{n-1}$

$$=\frac{1}{7}\{3^n+11(-4)^{n-1}\};$$

$$\therefore\ S_n=\frac{1}{7}\left\{\frac{3^{n+1}-3}{2}+11\frac{(-4)^n-1}{-5}\right\}=\frac{3^{n+1}}{14}-\frac{11}{35}(-4)^n+\frac{1}{10}.$$

(iii) Here $7+2p+q=0$ and $20+7p+2q=0$; $\therefore\ p=-2$ and $q=-3$.

Hence the generating function

$$=\frac{1+(2-2\,.\,1)x}{1-2x-3x^2}=\frac{1}{(1-3x)(1+x)}=\frac{\frac{3}{4}}{1-3x}+\frac{\frac{1}{4}}{1+x}.$$

Hence the nth term

$$=\frac{3}{4}\cdot 3^{n-1}+\frac{1}{4}(-1)^{n-1}=\frac{1}{4}\{3^n-(-1)^n\};$$

$$\therefore\ S_n=\frac{1}{4}\left\{\frac{3^{n+1}-3}{2}-\frac{(-1)^{n+1}+1}{-2}\right\}=\frac{1}{8}\{3^{n+1}-(-1)^n-2\}.$$

5. The scale of relation is $1-x-x^2$.

Hence the generating function

$$=\frac{1+(3-1)x}{1-x-x^2}=\frac{1+2x}{(1-\alpha x)(1-\beta x)}, \text{ where } \alpha=\frac{1+\sqrt{5}}{2} \text{ and } \beta=\frac{1-\sqrt{5}}{2}$$

$$=\frac{\frac{2+\alpha}{\alpha-\beta}}{1-\alpha x}+\frac{\frac{2+\beta}{\beta-\alpha}}{1-\beta x}.$$

Hence the nth term $=\frac{2+\alpha}{\alpha-\beta}\alpha^{n-1}+\frac{2+\beta}{\beta-\alpha}\beta^{n-1}$

$$=\frac{\sqrt{5}+1}{2}\alpha^{n-1}+\frac{1-\sqrt{5}}{2}\beta^{n-1}=\frac{1}{2^n}\{(1+\sqrt{5})^n+(1-\sqrt{5})^n\}.$$

6. Let $S = 1^3 + 2^3x + 3^3x^2 + \ldots\ldots$

By Art. 334 this is a recurring series whose scale of relation is $(1-x)^4$.

$\therefore$ the generating function

$$= \frac{a_0 + (a_1 + pa_0)\,x + (a_2 + pa_1 + qa_0)\,x^2 + (a_3 + pa_2 + qa_1 + a_0)\,x^3}{(1-x)^4}$$

$$= \frac{1^3 + (2^3 - 4 \,.\, 1^3)\,x + (3^3 - 4 \,.\, 2^3 + 6 \,.\, 1^3)\,x^2 + (4^3 - 4 \,.\, 3^3 + 6 \,.\, 2^3 - 4 \,.\, 1^3)\,x^3}{(1-x)^4}$$

$$= \frac{1 + 4x + x^2}{(1-x)^4}.$$

Hence $a = 1,\ b = 4,\ c = 1,\ d = 0.$

7. The first part is proved in Art. 334.

We thus have

$$a_0 + a_1x + \ldots\ldots + a_rx^r = (1-x)^{r+1}\,(1^r + 2^rx + 3^rx^2 + \ldots\ldots).$$

Equate the coefficients of x^r; then

$$a_r = (r+1)^r - (r+1)\,r^r + \frac{(r+1)\,r}{1 \,.\, 2}(r-1)^r - \ldots\ldots + (-1)^r\,(r+1)\,1^r$$

$$= 0, \text{ by Art. 305.}$$

Also $a_{r-s} = (r-s+1)^r - (r+1)\,(r-s)^r + \ldots\ldots + (-1)^{r-s}\,\frac{\lfloor r+1}{\underline{\lfloor r-s}\ \underline{\lfloor s+1}}\,1^r,$

and $a_{s-1} = s^r - (r+1)\,(s-1)^r + \ldots\ldots + (-1)^{s-1}\,\frac{\lfloor r+1}{\underline{\lfloor r-s+2}\ \underline{\lfloor s-1}}\,1^r.$

Hence

$$a_{r-s} - a_{s-1} = \left[(r-s+1)^r - (r+1)\,(r-s)^r + \ldots + (-1)^{r-s}\,\frac{\lfloor r+1}{\underline{\lfloor r-s}\ \underline{\lfloor s+1}}\,1^r\right]$$

$$+ (-1)^{r-s+1}\,\frac{\lfloor r+1}{\underline{\lfloor r-s+1}\ \underline{\lfloor s}}\,0^r + \left[(-1)^{r-s+2}\,\frac{\lfloor r+1}{\underline{\lfloor r-s+2}\ \underline{\lfloor s-1}}(-1)^r + \ldots + (-1)^{r+1}(-s)^r\right]$$

$$= 0, \text{ from Art. 305.}$$

Hence $a_{r-s} = a_{s-1}.$

8. We have

$$15+9p+5q+2r=0,\ 25+15p+9q+5r=0,\ \text{and}\ 43+25p+15q+9r=0;$$

$$\therefore p=-4,\ q=5\ \text{and}\ r=-2.$$

Hence the generating function

$$=\frac{2+(5-4\,.\,2)\,x+(9-4\,.\,5+5\,.\,2)\,x^2}{1-4x+5x^2-2x^3}$$

$$=\frac{2-3x-x^2}{(1-x)^2\,(1-2x)}=\frac{2}{(1-x)^2}-\frac{1}{1-x}+\frac{1}{1-2x}$$

$$=2\,(1+2x+3x^2+\ldots)-(1+x+x^2+\ldots)+(1+2x+2^2x^2+\ldots).$$

$\therefore$ the $(n+1)$th term

$$=\{2\,(n+1)-1+2^n\}\,x^n=\{2n+1+2^n\}\,x^n.$$

9. Here

$$26+11p+4q+r=0,\ 57+26p+11q+4r=0,\ \text{and}\ 120+57p+26q+11r=0;$$

$$\therefore p=-4,\ q=5\ \text{and}\ r=-2.$$

Hence, provided the series is convergent, the sum to infinity

$$=\frac{1+(4-4\,.\,1)\,x+(11-4\,.\,4+5\,.\,1)\,x^2}{(1-4x+5x^2-2x^3)}=\frac{1}{(1-x)^2\,(1-2x)}.$$

Since $2x<1$ the expansion of $(1-x)^{-2}\,(1-2x)^{-1}$ will be convergent.

10.
$$1+\frac{x}{a}+\frac{x\,(x+a)}{ab}+\frac{x\,(x+a)\,(x+b)}{abc}+\ldots$$

$$=\frac{x+a}{a}+\frac{x\,(x+a)}{ab}+\frac{x\,(x+a)\,(x+b)}{abc}+\ldots$$

$$=\frac{x+a}{a}\left(1+\frac{x}{b}+\frac{x\,(x+b)}{bc}+\ldots\right)$$

$$=\frac{(x+a)\,(x+b)}{ab}\left(1+\frac{x}{c}+\frac{x\,(x+c)}{cd}+\ldots\right)$$

$$=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=\frac{(x+a)\,(x+b)\ldots(x+l)}{abc\ldots l}.$$

11.
$$\frac{1}{x}-S=\frac{1}{x}-\frac{1}{x+a}-\frac{a}{(x+a)\,(x+b)}-\frac{ab}{(x+a)\,(x+b)\,(x+c)}-\ldots$$

$$=\frac{a}{x\,(x+a)}-\frac{a}{(x+a)\,(x+b)}-\frac{ab}{(x+a)\,(x+b)\,(x+c)}-\ldots$$

$$= \frac{a}{x+a}\left\{\frac{1}{x} - \frac{1}{x+b} - \frac{b}{(x+b)(x+c)} - \dots\right\}$$

$$= \frac{a}{x+a}\cdot\frac{b}{x+b}\left\{\frac{1}{x} - \frac{1}{x+c} - \frac{c}{(x+c)(x+d)} - \dots\right\}$$

$$= \dots = \frac{abc\dots l}{x(x+a)(x+b)\dots(x+l)}.$$

Hence $$S = \frac{1}{x} - \frac{abc\dots l}{x(x+a)(x+b)\dots(x+l)}.$$

12. $$\frac{1}{p+n} = \frac{1}{p-1}\left\{1 - \frac{1+n}{p+n}\right\},$$

$$\frac{1+n}{(p+n)(p+2n)} = \frac{1}{p-1}\left\{\frac{1+n}{p+n} - \frac{(1+n)(1+2n)}{(p+n)(p+2n)}\right\},$$

$$\frac{(1+n)(1+2n)}{(p+n)(p+2n)(p+3n)} = \frac{1}{p-1}\left\{\frac{(1+n)(1+2n)}{(p+n)(p+2n)} - \frac{(1+n)(1+2n)(1+3n)}{(p+n)(p+2n)(p+3n)}\right\}$$

$$\dots\dots\dots\dots = \dots\dots\dots\dots$$

Hence $$S_r = \frac{1}{p-1}\left\{1 - \frac{(1+n)(1+2n)\dots(1+rn)}{(p+n)(p+2n)\dots(p+rn)}\right\}.$$

Since $n+p$ is positive and $p>1$, $\dfrac{(1+n)(1+2n)\dots(1+rn)}{(p+n)(p+2n)\dots(p+rn)}$ is zero when r is infinite. [Art. 337, Ex. 1.]

Hence $$S_\infty = \frac{1}{p-1}.$$

13. We have $$\frac{1}{m+1} = \frac{1}{m-1}\left\{\frac{1}{1} - \frac{1.2}{m+1}\right\},$$

$$\frac{1.2}{(m+1)(m+2)} = \frac{1}{m-1}\left\{\frac{1.2}{m+1} - \frac{1.2.3}{(m+1)(m+2)}\right\},$$

$$\dots\dots\dots\dots = \dots\dots\dots\dots$$

Hence $$S_n = 1 + \frac{1}{m-1}\left\{\frac{1}{1} - \frac{1.2.3\dots n}{(m+1)(m+2)\dots(m+n-1)}\right\}.$$

Now if $m>1$, $\dfrac{2.3\dots n}{(m+1)(m+2)\dots(m+n-1)} = 0$ when n is infinite. [Art. 337, Ex. 1.]

Hence $$S_\infty = 1 + \frac{1}{m-1}.$$

14. We have $$\frac{1}{m+1}=\frac{1}{m+n}\left\{1+\frac{n-1}{m+1}\right\}.$$

$$-\frac{n-1}{(m+1)(m+2)}=-\frac{1}{m+n}\left\{\frac{n-1}{m+1}+\frac{(n-1)(n-2)}{(m+1)(m+2)}\right\},$$

$$\frac{(n-1)(n-2)}{(m+1)(m+2)(m+3)}=\frac{1}{m+n}\left\{\frac{(n-1)(n-2)}{(m+1)(m+2)}+\frac{(n-1)(n-2)(n-3)}{(m+1)(m+2)(m+3)}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$

Hence $$S_r=\frac{1}{m+n}\left\{1+(-1)^{r-1}\frac{(n-1)(n-2)\ldots(n-r)}{(m+1)(m+2)\ldots(m+r)}\right\}.$$

Now $(n-1)(n-2)\ldots(n-r)$ clearly vanishes when r is infinity provided n is a positive integer. Also, if n be not a positive integer and if $n<s$, where s is a positive integer; then if $m+n>0$,

$$\frac{(m+1)(m+2)\ldots(m+s)\ldots\text{to infinity}}{(1-n)(2-n)\ldots(s-n)\ldots\text{to infinity}}$$

$$=\frac{(m+1)\ldots(m+s-1)}{(1-n)\ldots(s-1-n)}\left(1+\frac{m+n}{s-n}\right)\left(1+\frac{m+n}{s+1-n}\right)\ldots$$

$$>\frac{(m+1)\ldots(m+s-1)}{(1-n)\ldots(s-1-n)}\left\{1+(m+n)\left(\frac{1}{s-n}+\frac{1}{s+1-n}+\ldots\text{to infinity}\right)\right\}.$$

But $\dfrac{1}{s-n}+\dfrac{1}{s+1-n}+\ldots$ to infinity is infinite, and therefore

$$\frac{(1-n)(2-n)\ldots\text{to infinity}}{(m+1)(m+2)\ldots\text{to infinity}}\text{ is zero when } m+n>0.$$

Hence $$S_\infty=\frac{1}{m+n}.$$

15. As in 14,

$$\frac{1}{n}.S_r=\frac{1}{n+1}-\frac{n-1}{(n+1)(n+2)}+\ldots+(-1)^{r-1}\frac{(n-1)\ldots(n-r+1)}{(n+1)\ldots(n+r)}$$

$$=\frac{1}{2n}\left\{1+(-1)^{r-1}\frac{(n-1)(n-2)\ldots(n-r)}{(n+1)(n+2)\ldots(n+r)}\right\}.$$

Hence $$\frac{1}{n}.S_n=\frac{1}{2n}\left\{1+(-1)^{n-1}\frac{(n-1)(n-2)\ldots(n-n)}{(n+1)(n+2)\ldots 2n}\right\}=\frac{1}{2n}.$$

16. It is obvious that $$1-\frac{2n+1}{2n+2}=\frac{1}{2n+2};$$

and $$1-2\frac{2n+1}{2n+2}+\frac{(2n+1)(2n+3)}{(2n+2)(2n+4)}=\frac{1.3}{(2n+2)(2n+4)}.$$

Hence the theorem is true for all values of n when $m=1$ and when $m=2$.

Assume then that

$$1-{}_mC_1\frac{2n+1}{2n+2}+\ldots+(-1)^r{}_mC_r\frac{(2n+1)\ldots(2n+2r-1)}{(2n+2)\ldots(2n+2r)}+\ldots$$

$$+(-1)^m{}_mC_m\frac{(2n+1)\ldots(2n+2m-1)}{(2n+2)\ldots(2n+2m)}=\frac{1.3.5\ldots(2m-1)}{(2n+2)(2n+4)\ldots(2n+2m)}$$

for all values of n and for any particular value of m.

Change n into $n+1$; we then have

$$1-{}_mC_1\frac{2n+3}{2n+4}+\ldots+(-1)^r{}_mC_r\frac{(2n+3)\ldots(2n+2r+1)}{(2n+4)\ldots(2n+2r+2)}+\ldots$$

$$+(-1)^m{}_mC_m\frac{(2n+3)\ldots(2n+2m+1)}{(2n+4)\ldots(2n+2m+2)}=\frac{1.3\ldots(2m-1)}{(2n+4)(2n+6)\ldots(2n+2m+2)}.$$

Multiply the latter series by $\frac{2n+1}{2n+2}$ and subtract from the former; then, since ${}_mC_r+{}_mC_{r-1}={}_{m+1}C_r$, we have

$$1-{}_{m+1}C_1\frac{2n+1}{2n+2}+\ldots+(-1)^r{}_{m+1}C_r\frac{(2n+1)\ldots(2n+2r-1)}{(2n+2)\ldots(2n+2r)}+\ldots$$

$$+(-1)^{m+1}{}_{m+1}C_{m+1}\frac{(2n+1)\ldots(2n+2m+1)}{(2n+2)\ldots(2n+2m+2)}$$

$$=\frac{1.3.5\ldots(2m-1)}{(2n+2)\ldots(2n+2m)}-\frac{2n+1}{2n+2}\frac{1.3.5\ldots(2m-1)}{(2n+4)\ldots(2n+2m+2)}$$

$$=\frac{1.3.5\ldots(2m-1)(2m+1)}{(2n+2)(2n+4)\ldots(2n+2m+2)}.$$

Thus if the theorem be true for all values of n and for any particular integral value of m, it will be true for the next greater value of m. Hence, as the theorem is true when $m=1$, it holds good for all integral values of m.

17. From Vandermonde's Theorem [Art. 249], we have

$$(a+b)_n=(a)_n+n(a)_{n-1}(b)_1+\frac{n(n-1)}{1.2}(a)_{n-2}(b)_2+\ldots$$

$$+\frac{\lfloor n}{\lfloor r\ \lfloor n-r}(a)_{n-r}(b)_r+\ldots$$

Hence $$\frac{(a+b)_n}{(a)_n} = 1 + n\frac{b}{a-n+1} + \frac{n(n-1)}{1.2}\frac{b(b-1)}{(a-n+1)(a-n+2)} + \ldots$$

$$+ \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\frac{b(b-1)\ldots(b-r+1)}{(a-n+1)(a-n+2)\ldots(a-n+r)} + \ldots .$$

Put $b = a = m$; and we have

$$1 + n\frac{m}{m-n+1} + \ldots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\frac{m(m-1)\ldots(m-r+1)}{(m-n+1)\ldots(m-n+r)} + \ldots$$

to $(n+1)$ terms $$= \frac{2m(2m-1)\ldots(2m-n+1)}{m(m-1)\ldots(m-n+1)}.$$

18. $$S = \left(1-\frac{1}{2}\right) - (3-1)\frac{m}{\underline{|3}} + (4-1)\frac{m(m-1)}{\underline{|4}} - (5-1)\frac{m(m-1)(m-2)}{\underline{|5}} + \ldots$$

$$= 1 - \frac{m}{\underline{|2}} + \frac{m(m-1)}{\underline{|3}} - \frac{m(m-1)(m-2)}{\underline{|4}} + \ldots$$

$$- \left\{\frac{1}{\underline{|2}} - \frac{m}{\underline{|3}} + \frac{m(m-1)}{\underline{|4}} - \frac{m(m-1)(m-2)}{\underline{|5}} + \ldots\right.$$

$$= \frac{1}{m+1} - \frac{1}{m+1}(1-1)^{m+1} + \left\{\frac{1}{(m+1)(m+2)} - \frac{1}{m+1}\right.$$

$$\left. - \frac{1}{(m+1)(m+2)}(1-1)^{m+2}\right\},$$

provided $m+1>0$ [Art. 338]

$$= \frac{1}{(m+1)(m+2)}.$$

19. $1.3.5\ldots(2n+1) = (1+2)(1+4)\ldots(1+2n) = 1 + P_1 + P_2 + \ldots + P_n.$

Also $2.4.6\ldots 2n = (1+1)(1+3)(1+5)\ldots(1+\overline{2n-1}) = 1 + Q_1 + Q_2 + \ldots + Q_n.$

20. $$\{a + (a+1) + (a+2) + \ldots + (a+n)\}\{a^2 + (a+1) + (a+2) + \ldots + (a+n)\}$$

$$= \left\{(n+1)a + \frac{1}{2}n(n+1)\right\}\left\{a^2 + na + \frac{1}{2}n(n+1)\right\}$$

$$= (n+1)a^3 + \frac{3}{2}n(n+1)a^2 + \frac{1}{2}n(n+1)(2n+1)a + \frac{1}{4}n^2(n+1)^2$$

$$= (n+1)a^3 + 3a^2(1+2+\ldots+n) + 3a(1^2+2^2+\ldots n^2) + 1^3 + 2^3 + \ldots + n^3$$

$$= a^3 + (a+1)^3 + \ldots + (a+n)^3.$$

21. Let

$$S_n = 1 - \frac{1-a^n}{1-a} + \frac{(1-a^n)(1-a^{n-1})}{(1-a)(1-a^2)} - \frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})}{(1-a)(1-a^2)(1-a^3)} + \ldots\ldots$$

$$\therefore (1-a^{n+1})S_n = (1-a^{n+1}) - \frac{(1-a^{n+1})(1-a^n)}{1-a} + \frac{(1-a^{n+1})(1-a^n)(1-a^{n-1})}{(1-a)(1-a^2)}$$
$$- \frac{(1-a^{n+1})(1-a^n)(1-a^{n-1})(1-a^{n-2})}{(1-a)(1-a^2)(1-a^3)} + \ldots\ldots$$

$$= \left(1 - \frac{a^{n+1}-a^{n+2}}{1-a}\right) - \frac{1-a^{n+1}}{1-a}\left(1 - \frac{a^n - a^{n+2}}{1-a^2}\right)$$
$$+ \frac{(1-a^{n+1})(1-a^n)}{(1-a)(1-a^2)}\left(1 - \frac{a^{n-1}-a^{n+2}}{1-a^3}\right)$$
$$- \frac{(1-a^{n+1})(1-a^n)(1-a^{n-1})}{(1-a)(1-a^2)(1-a^3)}\left(1 - \frac{a^{n-2}-a^{n+2}}{1-a^4}\right) + \ldots$$

$$= 1 - \frac{1-a^{n+2}}{1-a} + \frac{(1-a^{n+2})(1-a^{n+1})}{(1-a)(1-a^2)} - \frac{(1-a^{n+2})(1-a^{n+1})(1-a^n)}{(1-a)(1-a^2)(1-a^3)} + \ldots$$

$$= S_{n+2}.$$

Hence $S_{n+2} = (1-a^{n+1})S_n = (1-a^{n+1})(1-a^{n-1})(1-a^{n-3})\ldots$

If n is odd the last factor is $1-a^0 = 0$. If n is even the last factor is $1-a$.

Or thus:

We have from Examples XXIX. 30

$$\frac{1}{(1-x)(1-ax)(1-a^2x)\ldots} = 1 + \frac{x}{1-a} + \frac{x^2}{(1-a)(1-a^2)} + \ldots$$

Hence $$\frac{1}{(1+x)(1+ax)(1+a^2x)\ldots} = 1 - \frac{x}{1-a} + \frac{x^2}{(1-a)(1-a^2)} - \ldots,$$

and $$\frac{1}{(1-x^2)(1-a^2x^2)(1-a^4x^2)\ldots} = 1 + \frac{x^2}{1-a^2} + \frac{x^4}{(1-a^2)(1-a^4)} + \ldots$$

Hence $$\left\{1 + \frac{x}{1-a} + \frac{x^2}{(1-a)(1-a^2)} + \ldots\right\}\left\{1 - \frac{x}{1-a} + \frac{x^2}{(1-a)(1-a^2)} - \ldots\right\}$$
$$= 1 + \frac{x^2}{1-a^2} + \frac{x^4}{(1-a^2)(1-a^4)} + \ldots$$

Equate the coefficients of x^n on the two sides of the last equation and the result follows at once.

[Heine, Crelle t. XXXIX.]

22. $$S=\frac{n}{1\cdot2\cdot3}+\frac{n-1}{2\cdot3\cdot4}+\frac{n-2}{3\cdot4\cdot5}+\dots+\frac{1}{n(n+1)(n+2)}$$

$$=\frac{(n+1)-1}{1\cdot2\cdot3}+\frac{(n+1)-2}{2\cdot3\cdot4}+\frac{(n+1)-3}{3\cdot4\cdot5}+\dots+\frac{(n+1)-n}{n(n+1)(n+2)}$$

$$=(n+1)\left\{\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\dots+\frac{1}{n(n+1)(n+2)}\right\}$$

$$-\left\{\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\dots+\frac{1}{(n+1)(n+2)}\right\}$$

$$=(n+1)\cdot\frac{1}{2}\left\{\frac{1}{1\cdot2}-\frac{1}{(n+1)(n+2)}\right\}-\left\{\frac{1}{2}-\frac{1}{n+2}\right\}=\frac{n(n+1)}{4(n+2)}.$$

23. $$2S=\frac{(1+3)x^2}{1\cdot3}-\frac{(2+4)x^3}{2\cdot4}+\frac{(3+5)x^4}{3\cdot5}-\dots$$

$$=\left\{x^2-\frac{x^3}{2}+\frac{x^4}{3}-\dots\right\}+\left\{\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\dots\right\}$$

$$=x\log(1+x)+\frac{1}{x}\log(1+x)-1+\frac{x}{2}.$$

24. $$S=\frac{x}{1\cdot2}+\frac{x^2}{2\cdot3}+\dots+\frac{x^n}{n(n+1)}+\dots$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)x+\left(\frac{1}{2}-\frac{1}{3}\right)x^2+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)x^n+\dots$$

$$=\left\{x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^n}{n}+\dots\right\}-\left\{\frac{x}{2}+\frac{x^2}{3}+\dots+\frac{x^n}{n+1}+\dots\right\}$$

$$=-\log(1-x)+\frac{1}{x}\log(1-x)+1.$$

25. Let $S=1\cdot2\cdot3+3\cdot4\cdot5x+5\cdot6\cdot7x^2+\dots$

Then, x being less than unity,

$$S(1-x)^4=1\cdot2\cdot3+(3\cdot4\cdot5-4\cdot1\cdot2\cdot3)x$$
$$+(5\cdot6\cdot7-4\cdot3\cdot4\cdot5+6\cdot1\cdot2\cdot3)x^2$$
$$+(7\cdot8\cdot9-4\cdot5\cdot6\cdot7+6\cdot3\cdot4\cdot5-4\cdot1\cdot2\cdot3)x^3$$

the other terms all vanishing [Art. 326].

Hence $$S=\frac{6+36x+6x^2}{(1-x)^4}.$$

26. $$S=1-3n+\frac{3n(3n-3)}{1\cdot 2}-\frac{3n(3n-4)(3n-5)}{1\cdot 2\cdot 3}+\ldots$$
$$=1-(3n-1)+\frac{(3n-2)(3n-3)}{1\cdot 2}-\frac{(3n-3)(3n-4)(3n-5)}{1\cdot 2\cdot 3}+\ldots$$
$$-\left\{1-(3n-3)+\frac{(3n-4)(3n-5)}{1\cdot 2}-\ldots\right\}.$$

Now $(1+x)(1+x^3)^{-1}=(1-x+x^2)^{-1}=1+x(1-x)+x^2(1-x)^2+\ldots$

$$\therefore\ (1+x)(1-x^3+x^6-\ldots)=1+x+\ldots+x^r\left\{1-(r-1)+\frac{(r-2)(r-3)}{1\cdot 2}-\ldots\right\}+\ldots$$

Equating the coefficients of x^{3n} and x^{3n-2} on the two sides of the last equation, we have

$$(-1)^n=1-(3n-1)+\frac{(3n-2)(3n-3)}{1\cdot 2}-\ldots,$$

and
$$-(-1)^n=1-(3n-3)+\frac{(3n-4)(3n-5)}{1\cdot 2}+\ldots$$

27. $$S_n=\left\{1-\frac{1}{a_1+1}\right\}+\left\{\frac{1}{a_1+1}-\frac{1}{(a_1+1)(a_2+1)}\right\}+\ldots$$
$$+\left\{\frac{1}{(a_1+1)(a_2+1)\ldots(a_{n-1}+1)}-\frac{1}{(a_1+1)(a_2+1)\ldots(a_n+1)}\right\}$$
$$=1-\frac{1}{(a_1+1)(a_2+1)\ldots(a_n+1)}.$$

Now $(a_1+1)(a_2+1)\ldots(a_n+1)$ is greater than $a_1+a_2+\ldots+a_n$. Hence, as $a_1+a_2+a_3+\ldots$ is divergent, $S_\infty=1$.

28. $$u_n=\frac{n^m}{(n+1)^{m+x}}=\frac{1}{(n+1)^x\left(1+\frac{1}{n}\right)^m}.$$

Now $\left(1+\frac{1}{n}\right)^m$ is finite and lies between 1^m and 2^m.

Hence, by Art. 270, the given series is convergent or divergent according as the series whose nth term is $\frac{1}{(n+1)^x}$ is convergent or divergent, that is according as x is greater or not greater than unity.

29. Let the series be $v_1+v_2+v_3+\ldots$

Then
$$1+v_1=\frac{u_1+u_2}{u_1},\ 1+v_2=\frac{u_1+u_2+u_3}{u_1+u_2},\ldots,$$
$$1+v_n=\frac{u_1+u_2+\ldots+u_{n+1}}{u_1+u_2+\ldots+u_n},\ldots$$

Hence $$(1+v_1)(1+v_2)(1+v_3)\ldots = \frac{u_1+u_2+u_3+\ldots}{u_1}.$$

Hence, from Art. 337, Σv_r is convergent or divergent according as Σu_r is convergent or divergent.

30. $(1+a)(1+ax)(1+ax^2)(1+ax^3)\ldots$ is convergent if

$$a+ax+ax^2+ax^3+\ldots$$

is convergent; that is, if $x<1$.

31. Since $v_1>v_2>v_3>v_4\ldots$, we have

$$v_1v_2v_3v_4\ldots < v_1v_2^2v_4^4\ldots;$$

and also $$(v_1v_2v_3v_4\ldots)^2 > v_1\cdot v_1v_2^2v_4^4\ldots.$$

Hence $v_1v_2v_3v_4\ldots$ and $v_1v_2^2v_4^4\ldots$ are both finite or both infinite.

32. (i) $$\frac{n^n}{(n+1)^{n+1}} = \frac{1}{n+1}\frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

Now $\left(1+\frac{1}{n}\right)^n$ increases continuously as n is increased, but $\left(1+\frac{1}{n}\right)^n$ never exceeds the finite quantity e.

Hence $$e\cdot\Sigma\frac{n^n}{(n+1)^{n+1}} > \Sigma\frac{1}{n+1}.$$

$\Sigma\frac{n^n}{(n+1)^{n+1}}$ is therefore divergent.

(ii) $\frac{1}{\sqrt[n]{n^{n+1}}} = \frac{1}{n\sqrt[n]{n}}$. Now $e^n = 1+n+\ldots$; therefore $e^n > n$ and therefore

$$e > n^{\frac{1}{n}}.$$

Hence $$\Sigma\frac{1}{\sqrt[n]{n^{n+1}}} > \frac{1}{e}\Sigma\frac{1}{n};$$

the series is therefore divergent.

(iii) Here $$n\left(1-\frac{u_{n+1}}{u_n}\right) = n\left\{1-\frac{(2n+2)^2}{(2n+3)(2n+4)}\right\}$$

$$= n\,\frac{6n+8}{(2n+3)(2n+4)} = \frac{3}{2} \text{ in the limit.}$$

Hence, from Art. 338, the series is convergent.

Or thus:

$$u_n = \frac{1}{2n+2} \cdot \frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)},$$

and [Art. 207]

$$u_n < \frac{1}{2n+2} \frac{3 \cdot 5 \cdot 7 \ldots (2n+1)}{4 \cdot 6 \cdot 8 \ldots (2n+2)}; \quad \therefore \; u_n < \frac{1}{(2n+2)^{\frac{3}{2}}}.$$

Hence $$\Sigma u_n < \Sigma \frac{1}{(2n+2)^{\frac{3}{2}}} < \Sigma \frac{1}{n^{\frac{3}{2}}}.$$

(iv) $$u_n = \frac{1}{2n+1} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n},$$

and $$u_n < \frac{1}{2n+1} \cdot \frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)}.$$

Hence $$u_n < \frac{1}{(2n+1)^{\frac{3}{2}}} < \frac{1}{n^{\frac{3}{2}}}.$$

(v) Here $$n\left(1 - \frac{u_{n+1}}{u_n}\right) = n\left\{1 - \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}\right\}$$

$$= n \frac{n(1+\gamma-\alpha-\beta)+\gamma-\alpha\beta}{n^2+n(\gamma+1)+\gamma} = 1+\gamma-\alpha-\beta \text{ in the limit.}$$

Hence the series is convergent if $1+\gamma-\alpha-\beta>1$, or $\gamma>\alpha+\beta$; it is divergent if $1+\gamma-\alpha-\beta<1$, or $\gamma<\alpha+\beta$.

If $1+\gamma-\alpha-\beta=0$, we have

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots(\alpha+n)\,\beta(\beta+1)\ldots(\beta+n)}{1 \cdot 2 \ldots (n+1)(\alpha+\beta)(\alpha+\beta+1)\ldots(\alpha+\beta+n)}.$$

Now $$\frac{(\beta+1)(\beta+2)\ldots(\beta+n)}{1 \cdot 2 \ldots n} > \frac{(\alpha+\beta)(\alpha+\beta+1)\ldots(\alpha+\beta+n-1)}{\alpha(\alpha+1)\ldots(\alpha+n-1)}.$$

Hence $$u_n > \frac{\beta(\alpha+n)}{(n+1)(\alpha+\beta+n)}.$$

But $\frac{\beta(\alpha+n)}{(n+1)(\alpha+\beta+n)} > \frac{\beta}{2n}$, either from the beginning or after a finite number of terms. Hence, as $\Sigma \frac{\beta}{2n}$ is divergent, Σu_n is divergent.

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1. $y^2z^2+z^2x^2+x^2y^2=\frac{1}{2}x^2(y^2+z^2)+\frac{1}{2}y^2(z^2+x^2)+\frac{1}{2}z^2(x^2+y^2)$

$>x^2yz+y^2zx+z^2xy$ [unless $x=y=z$]

$>xyz\,(x+y+z)$.

2. $(a_1^2+a_2^2+\ldots)(b_1^2+b_2^2+\ldots)=a_1^2b_1^2+a_2^2b_2^2+\ldots+\Sigma(a_r^2b_s^2+a_s^2b_r^2)$

$\geqslant\Sigma a_1^2b_1^2+2\Sigma a_ra_sb_sb_r$, since $a_r^2b_s^2+a_s^2b_r^2\geqslant 2a_ra_sb_rb_s$

$\geqslant(a_1b_1+a_2b_2+\ldots)^2$.

3. $\Sigma a_1b_1\,.\,\Sigma\frac{a_1}{b_1}=\Sigma a_1^2+\Sigma\left(a_rb_r\,.\,\frac{a_s}{b_s}+a_sb_s\,\frac{a_r}{b_r}\right)$

$=\Sigma a_1^2+\Sigma a_ra_s\left(\frac{b_r}{b_s}+\frac{b_s}{b_r}\right)$

$\geqslant\Sigma a_1^2+2\Sigma a_ra_s\left[\text{since }\frac{b_r}{b_s}+\frac{b_s}{b_r}\geqslant 2\right]$

$\geqslant(\Sigma a_1)^2$.

4. $(a^5-b^5)(a-b)>0$, since *both* factors are positive or *both* negative.

Hence $a^6+b^6>a^5b+ab^5$.

5. Unless $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}>3\sqrt[3]{\frac{xyz}{abc}}$,

and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}>3\sqrt[3]{\frac{abc}{xyz}}$.

Hence $\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)>9$.

6. $a+b+c>\sqrt[3]{abc}$ and $a^2+b^2+c^2>3\sqrt[3]{a^2b^2c^2}$;

$\therefore\ (a+b+c)(a^2+b^2+c^2)>9abc$.

7. $a^2cd+b^2ad+c^2ab+d^2bc$

$>4\sqrt[4]{a^2cd\,.\,b^2ad\,.\,c^2ab\,.\,d^2bc}>4abcd$, unless $a=b=c=d$.

8. $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)$

$=\frac{1}{2}a^2(b^2+c^2)+\frac{1}{2}b^2(c^2+a^2)+\frac{1}{2}c^2(a^2+b^2)+2abc(a+b+c)$

$\geqslant a^2bc+b^2ca+c^2ab+2abc(a+b+c)\geqslant 3abc(a+b+c)$.

9. $b^4+c^4>2b^2c^2$, $c^4+a^4>2c^2a^2$ and $a^4+b^4>2a^2b^2$;

$$\therefore\ 2(a^4+b^4+c^4)>2b^2c^2+2c^2a^2+2a^2b^2$$
$$>a^2(b^2+c^2)+b^2(c^2+a^2)+c^2(a^2+b^2)$$
$$>a^2\cdot 2bc+b^2\cdot 2ca+c^2\cdot 2ab$$
$$>2abc(a+b+c).$$

10. Unless $a=b=c=d$, $\dfrac{a^5+b^5+c^5+d^5}{4}$

$$>\frac{a+b+c+d}{4}\cdot\left(\frac{a+b+c+d}{4}\right)^4 \text{ [Theorem V]},$$
$$>\frac{a+b+c+d}{4}\cdot abcd \text{ [Theorem II]}.$$

11. $\dfrac{a-x}{a+x}<\dfrac{a^2-x^2}{a^2+x^2}$, if $(a-x)(a^2+x^2)<(a+x)(a^2-x^2)$, if $2ax(x-a)<0$, if $x-a<0$.

12. $\dfrac{1}{b}+\dfrac{1}{c}\not<\dfrac{2}{\sqrt{bc}}$, $\dfrac{1}{c}+\dfrac{1}{a}\not<\dfrac{2}{\sqrt{ca}}$, $\dfrac{1}{a}+\dfrac{1}{b}\not<\dfrac{2}{\sqrt{ab}}$;

$$\therefore\ 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\not<2\left(\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}\right).$$

13. $\dfrac{X_1^2+X_2^2+\ldots+X_n^2}{n}>\left(\dfrac{X_1+X_2+\ldots+X_n}{n}\right)^2$;

$\therefore\ n\Sigma X_1^2>(\Sigma X_1)^2$. Also $(\Sigma X_1)^2=\Sigma X_1^2+2\Sigma X_1X_2>\Sigma X_1^2$.

14. Put $x_1=a+y_1$, &c. Then we have $y_1y_2\ldots y_n=b^n$, and we require the least value of $(a+y_1)(a+y_2)\ldots(a+y_n)$.

Now if y_r and y_s, any two of the quantities y_1, y_2, &c. are unequal, and if $y_ry_s=y^2$, then $y_r+y_s>2\sqrt{y_ry_s}>2y$. Hence

$$(a+y_r)(a+y_s)=a^2+y^2+a(y_r+y_s)>(a+y)^2.$$

Hence if any two of the y's are unequal, the product $(a+y_1)(a+y_2)\ldots(a+y_n)$ would be diminished by removing these two and substituting for each their geometric mean. The product must therefore be least when all the y's are equal, and therefore all equal to b; and therefore the least value of the product is $(a+b)^n$.

15. $x^2+y^2\not<2xy$; $\therefore\ ab(x^2+y^2)\not<2abxy$;

$$\therefore\ (a^2+b^2)xy+ab(x^2+y^2)\not<(a^2+b^2+2ab)xy,$$

that is
$$(ax+by)(ay+bx)\not<(a+b)^2xy;$$
$$\therefore\ \frac{(a+b)xy}{ay+bx}\not>\frac{ax+by}{a+b}.$$

16. Unless $a=b=c$, $\dfrac{\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}}{3} > \sqrt[3]{\dfrac{8}{(b+c)(c+a)(a+b)}}$,

and $\dfrac{(b+c)+(c+a)+(a+b)}{3} > \sqrt[3]{(b+c)(c+a)(a+b)}$.

Hence $\dfrac{1}{3}\left(\dfrac{2}{b+c}+\dfrac{2}{c+a}+\dfrac{2}{a+b}\right)\dfrac{2(a+b+c)}{3} > 2$,

or $\dfrac{2}{b+c}+\dfrac{2}{c+a}+\dfrac{2}{a+b} > \dfrac{9}{a+b+c}$.

17. Unless $a=b=c=d$,

$$\frac{\frac{3}{b+c+d}+\dots}{4} > \frac{3}{\sqrt[4]{\{(b+c+d)(c+d+a)(d+a+b)(a+b+c)\}}}, \text{ and}$$

$$\frac{3(a+b+c+d)}{4} = \frac{\Sigma(b+c+d)}{4} > \sqrt[4]{\{(b+c+d)(c+d+a)(d+a+b)(a+b+c)\}};$$

$$\therefore \quad \frac{\frac{3}{b+c+d}+\dots}{4} \times \frac{3(a+b+c+d)}{4} > 3.$$

18. If $a>b>c$; then $\left(\dfrac{a+c}{a-c}\right)^a < \left(\dfrac{b+c}{b-c}\right)^b$,

if $a\left[\log\left(1+\dfrac{c}{a}\right)-\log\left(1-\dfrac{c}{a}\right)\right] < b\left[\log\left(1+\dfrac{c}{b}\right)-\log\left(1-\dfrac{c}{b}\right)\right]$,

i.e. if $a\left(\dfrac{c}{a}+\dfrac{1}{3}\dfrac{c^3}{a^3}+\dots\right) < b\left(\dfrac{c}{b}+\dfrac{1}{3}\dfrac{c^3}{b^3}+\dots\right)$

[the expansions of $\log\left(1\pm\dfrac{c}{a}\right)$ and $\log\left(1\pm\dfrac{c}{b}\right)$ being possible since $\dfrac{c}{a}$ and $\dfrac{c}{b}$ are both less than unity],

if $c+\dfrac{1}{3}\dfrac{c^3}{a^2}+\dfrac{1}{5}\dfrac{c^5}{a^4}+\dots < c+\dfrac{1}{3}\dfrac{c^3}{b^2}+\dfrac{1}{5}\dfrac{c^5}{b^4}+\dots$,

which is obvious since $a>b$.

19. If $x^2=y^2+z^2$, then $x>y$ and $x>z$.

Hence $x^{n-2}-y^{n-2} \gtrless 0$ according as $n-2 \gtrless 0$.

Also $x^{n-2}-z^{n-2} \gtrless 0$.....

Hence $x^2y^2(x^{n-2}-y^{n-2})+x^2z^2(x^{n-2}-z^{n-2}) \gtrless 0$, according as $n-2 \gtrless 0$;

$$\therefore\ x^n(y^2+z^2) \gtrless x^2(y^n+z^n).\ldots\ldots\ldots\ldots$$

$$\therefore\ x^n \gtrless y^n+z^n.\ldots\ldots\ldots\ldots$$

20. Let λ be the greatest of the four quantities $a^{\frac{1}{p}}, b^{\frac{1}{q}}, c^{\frac{1}{r}}, d^{\frac{1}{s}}$; then $abcd < \lambda^p \cdot \lambda^q \cdot \lambda^r \cdot \lambda^s < \lambda^{p+q+r+s}$.

Hence $$(abcd)^{\frac{1}{p+q+r+s}} < \lambda.$$

Similarly if μ be the smallest, we have

$$(abcd)^{\frac{1}{p+q+r+s}} > \mu.$$

21. $$\frac{1+x+x^2+\ldots+x^{2n}}{2n+1} > \sqrt[2n+1]{\{1 \cdot x \cdot x^2 \ldots x^{2n}\}};$$

$$\therefore\ 1+x+x^2+\ldots+x^{2n} > (2n+1)\,x^n.$$

22. We have to shew that $n(a^{2n+1}+1) > a^{2n}+a^{2n-1}+\ldots+a^2+a$.

Now $(a^{2n}-1)(a-1) > 0$, and therefore $a^{2n+1}+1 > a^{2n}+a$. Similarly $a^{2n+1}+1 > a^{2n-1}+a^2$; and so on.

Hence $2n(a^{2n+1}+1) > (a^{2n}+a)+(a^{2n-1}+a^2)+\ldots+(a+a^{2n})$.

23. $$(m+1)(m+2n-1) < \left\{\frac{(m+1)+(m+2n-1)}{2}\right\}^2 < (m+n)^2.$$

So $$(m+2)(m+2n-2) < (m+n)^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(m+2n-1)(m+1) < (m+n)^2.$$

Hence $$\{(m+1)(m+2)\ldots(m+2n-1)\}^2 < (m+n)^{2(2n-1)}.$$

24. $a^2 > a^2-(b-c)^2$, $b^2 > b^2-(c-a)^2$ and $c^2 > c^2-(a-b)^2$.

Hence $$a^2b^2c^2 > \{(b+c-a)(c+a-b)(a+b-c)\}^2;$$

$$\therefore\ abc > (b+c-a)(c+a-b)(a+b-c).$$

25. $$\sqrt[3]{\{(c+d+a-2b)(d+a+b-2c)(a+b+c-2d)\}}$$

$$< \frac{1}{3}\{(c+d+a-2b)+(d+a+b-2c)+(a+b+c-2d)\} < a.$$

So $$\sqrt[3]{\{(d+a+b-2c)(a+b+c-2d)(b+c+d-2a)\}} < b,$$

$$\sqrt[3]{\{(a+b+c-2d)(b+c+d-2a)(c+d+a-2b)\}} < c,$$

and $$\sqrt[3]{\{(b+c+d-2a)(c+d+a-2b)(d+a+b-2c)\}} < d.$$

Hence $(b+c+d-2a)(c+d+a-2b)(d+a+b-2c)(a+b+c-2d) < abcd$.

26. $\sqrt[n-1]{\{(s-a_2)(s-a_3)\ldots(s-a_n)\}} < \dfrac{(s-a_2)+(s-a_3)+\ldots+(s-a_n)}{n-1} < \dfrac{a_1}{n-1}$.

So $\sqrt[n-1]{\{(s-a_3)\ldots(s-a_n)(s-a_1)\}} < \dfrac{a_2}{n-1}$.

.....

$\sqrt[n-1]{\{(s-a_1)(s-a_2)\ldots(s-a_{n-1})\}} < \dfrac{a_n}{n-1}$.

Hence $(s-a_1)(s-a_2)\ldots(s-a_n) < \dfrac{a_1 a_2 \ldots a_n}{(n-1)^n}$.

27. $$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}$$
$$> \frac{3}{\sqrt[3]{\{(b+c-a)(c+a-b)(a+b-c)\}}}$$
$$> \frac{3}{\sqrt[6]{[\{a^2-(b-c)^2\}\{b^2-(c-a)^2\}\{c^2-(a-b)^2\}]}}$$
$$> \frac{3}{\sqrt[3]{abc}} > \frac{3}{\frac{a+b+c}{3}} > \frac{9}{a+b+c}.$$

28. Suppose that a, b and c are in order of magnitude; then the result is obvious unless $a > b+c$. Let $a = b+c+x$, where x is positive; then we have to shew that

$$-x(b-c)^2+(b+x)^2(2c+x)+(c+x)^2(2b+x)>0,$$

or
$$x\{b^2+c^2+8bc-(b-c)^2\}+\text{positive terms}>0,$$

that is
$$10bcx+\text{positive terms}>0.$$

29. Let $a>b>c$. The expansion on the left may be written

$$a^2(a-b)(a-c)-b^2(a-b)(a-c+b-a)-c^2(a-c)(a-b+c-a)$$
$$=(a-b)(a-c)(a^2-b^2-c^2)+(a-b)^2b^2+(a-c)^2c^2$$
$$=(a-b)(a-c)\{a^2-(b-c)^2\}+\{(a-b)b-(a-c)c\}^2,$$

which is obviously positive since $a>b>c$.

30. First suppose that p, q and r are integers. Then, unless

$$x^{q-r}=x^{p-q}=x^{r-p}$$

which requires that $x=1$ or else $p=q=r$, we have

$$\frac{(x^{q-r}+x^{q-r}+\ldots\text{to } p \text{ terms})+(x^{r-p}+x^{r-p}+\ldots\text{to } q \text{ terms})+(x^{p-q}+x^{p-q}+\ldots\text{to } r \text{ terms})}{p+q+r}$$
$$> \sqrt[p+q+r]{\{x^{p(q-r)}\cdot x^{q(r-p)}\cdot x^{r(p-q)}\}} > 1;$$
$$\therefore\ px^{q-r}+qx^{r-p}+rx^{p-q} > p+q+r.$$

But if p, q, r are not all integers let k be such that kp, kq, kr are all integers; then as above

$$\frac{kp \,.\, x^{q-r} + kq \,.\, x^{r-p} + kr \,.\, x^{q-p}}{kp+kq+kr} > \left\{x^{kp(q-r)} \,.\, x^{kq(r-p)} \,.\, x^{kr(p-q)}\right\}^{\frac{1}{kp+kq+kr}} > 1.$$

31. $$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}.$$

$$\therefore \left\{\frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{2 \,.\, 4 \,.\, 6 \ldots 2n}\right\}^2 < \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{2 \,.\, 4 \,.\, 6 \ldots 2n} \cdot \frac{2 \,.\, 4 \ldots 2n}{3 \,.\, 5 \ldots (2n+1)} < \frac{1}{2n+1}.$$

32. $$\frac{3}{5} \cdot \frac{7}{9} \cdot \frac{11}{13} \cdots \frac{4n-1}{4n+1} < \frac{5}{7} \cdot \frac{9}{11} \cdot \frac{13}{15} \cdots \frac{4n+1}{4n+3}.$$

Hence $$\left\{\frac{3 \,.\, 7 \,.\, 11 \ldots (4n-1)}{5 \,.\, 9 \,.\, 13 \ldots (4n+1)}\right\}^2 < \frac{3}{4n+3}.$$

33. Since the sum of ax^α, by^β and cz^γ is given, $ax^\alpha \,.\, by^\beta \,.\, cz^\gamma$ is greatest when $ax^\alpha = by^\beta = cz^\gamma$, and then each is equal to $\frac{d}{3}$. Hence the greatest value of $x^\alpha y^\beta z^\gamma$ is $\frac{1}{27} \frac{d^3}{abc}$.

34. It is easy to see that $1 \,.\, n = n$, $2\,(n-1) > n$,

$$3\,(n-2) > n, \ldots, \quad r\,(n-r+1) > n, \ldots, \quad n \,.\, 1 = n.$$

Hence $$(\lfloor n)^2 > n^n.$$

35. $$(1+x)^n > (2x^{\frac{1}{2}})^n$$

$$1 + x^n > 2x^{\frac{n}{2}};$$

$$\therefore \; (1+x)^n\,(1+x^n) > 2^{n+1} x^n.$$

36. We have to shew that

$$\frac{1+a^2+a^4+\ldots a^{2n}}{n+1} > \frac{a+a^3+\ldots+a^{2n-1}}{n}, \quad \text{or} \quad \frac{1+a^2+\ldots+a^{2n}}{a+a^3+\ldots+a^{2n-1}} > \frac{n+1}{n}.$$

Put $F(n)$ for $\frac{1+a^2+\ldots+a^{2n}}{a+a^3+\ldots+a^{2n-1}}$; then

$$F(n+1) + \frac{1}{F(n)} = \frac{1+a^2+\ldots+a^{2n+2}}{a+a^3+\ldots+a^{2n+1}} + \frac{a+a^3+\ldots+a^{2n-1}}{1+a^2+\ldots+a^{2n}}$$

$$= \frac{(1+a^2)\,(1+a^2+a^4+\ldots a^{2n})}{a\,(1+a^2+a^4+\ldots+a^{2n})} = a + \frac{1}{a}.$$

Now $a+\frac{1}{a}>2$; therefore $F(n+1)>2-\frac{1}{F(n)}$.

Hence, *if* $F(n)>\frac{n+1}{n}$, then $F(n+1)>2-\frac{n}{n+1}>\frac{n+2}{n+1}$.

But $F(1)=\frac{1+a^2}{a}>\frac{1+1}{1}$. Hence the theorem is true by induction.

37. Let a and ar^{n-1} be the first and last terms, and let there be n of them.

Then we have to prove that

$$\frac{n}{2}(a+ar^{n-1})>a+ar+\ldots+ar^{n-1}.$$

This follows at once from

$$1+r^{n-1}=1+r^{n-1},\quad 1+r^{n-1}>r+r^{n-2},\quad 1+r^{n-1}>r^2+r^{n-3},$$
$$\ldots,\quad 1+r^{n-1}>r^s+r^{n-s-1},\quad \ldots,\quad 1+r^{n-1}=r^{n-1}+1.$$

38.
$$P_r=\{\sqrt[r]{a_1a_2\ldots a_r}+\sqrt[r]{a_2a_3\ldots a_{r+1}}+\ldots\}/{}_nC_r.$$
$$P_{r-1}=\{\sqrt[r-1]{a_1a_2\ldots a_{r-1}}+\sqrt[r-1]{a_2a_3\ldots a_r}+\ldots\}/{}_nC_{r-1}.$$

Consider all the terms in P_{r-1} which contain r particular letters, say $a_1, a_2, \ldots a_r$. There are r of these terms and

$$\sqrt[r-1]{a_1a_2\ldots a_{r-1}}+\sqrt[r-1]{a_2a_3\ldots a_r}+\sqrt[r-1]{a_3a_4\ldots a_ra_1}+\ldots$$
$$>r\sqrt[r]{\left[\sqrt[r-1]{a_1a_2\ldots a_r}\cdot\sqrt[r-1]{a_2a_3\ldots a_r}\cdot\sqrt[r-1]{a_3a_4\ldots a_1}\ldots\right]}$$
$$>r\sqrt[r]{a_1a_2\ldots a_r}.$$

And by taking the terms of P_{r-1} in groups as above, each term of P_{r-1} would be taken $n-(r-1)$ times and each group would be greater than $r\times$ corresponding term of P_r.

Hence
$$(n-r+1)P_{r-1}>rP_r,$$

and therefore
$$\frac{P_{r-1}}{{}_nC_{r-1}}>\frac{P_r}{{}_nC_r}.$$

39. First, suppose that a, b, c are integers; then unless $a=b=c$,

$$\frac{\{(s-a)+(s-a)+\ldots\text{to } a \text{ terms}\}+\{(s-b)+(s-b)+\ldots\text{to } b \text{ terms}\}+\ldots}{a+b+\ldots}$$
$$>\sqrt[a+b+\ldots]{\{(s-a)^a(s-b)^b\ldots\ldots\}};$$
$$\therefore\quad \left\{\frac{\Sigma a(s-a)}{s}\right\}^s>(s-a)^a(s-b)^b\ldots\ldots \qquad \ldots\ldots\ldots\ldots \text{(i)}.$$

Again $$\Sigma a\,(s-a)=s^2-\Sigma a^2,$$

and $$s^2-\Sigma a^2<\frac{n-1}{n}s^2,$$

if $$n\Sigma a^2>(a+b+\ldots)^2,$$

that is, if $\Sigma\,(a-b)^2>0$, which is obvious.

And, since $$\Sigma a\,(s-a)<\frac{n-1}{n}s^2,$$

$$\left\{\frac{n-1}{n}s\right\}^s>\left\{\frac{\Sigma a\,(s-a)}{s}\right\}^s \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

From (i) and (ii) it follows that

$$\left(\frac{s}{n}\right)^s>\left(\frac{s-a}{n-1}\right)^a\left(\frac{s-b}{n-1}\right)^b\ldots\ldots$$

If a, b, ... be not all integral, let ka, kb, ... be all integral; then, by the above,

$$\left(\frac{ks}{n}\right)^{ks}>\left(\frac{ks-ka}{n-1}\right)^{ka}\left(\frac{ks-kb}{n-1}\right)^{kb}\ldots;$$

$$\therefore\quad\left(\frac{s}{n}\right)^s>\left(\frac{s-a}{n-1}\right)^a\left(\frac{s-b}{n-1}\right)^b\ldots$$

40. $$(1+1)^n=1+\frac{n}{1}+\frac{n\,(n-1)}{1\,.\,2}+\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}+\ldots+\frac{\lfloor n}{\lfloor n}.$$

The G. M. of the series on the right is

$$\sqrt[n+1]{\frac{n^n\,.\,(n-1)^{n-1}\,.\,(n-2)^{n-2}\ldots 2^2\,.\,1}{1^n\,.\,2^{n-1}\,.\,3^{n-2}\ldots(n-1)^2\,.\,n^1}}.$$

The A. M. is $\frac{2^n}{n+1}$.

Hence $$2^{n(n+1)}>(n+1)^{n+1}\left(\frac{n}{1}\right)^n\left(\frac{n-1}{2}\right)^{n-1}\ldots\left(\frac{2}{n-1}\right)^2\left(\frac{1}{n}\right).$$

XXXVI. Pages 454–456.

1. By Art. 352, $$p_3=\lambda p_2+p_1,\quad q_3=\lambda q_2+q_1;$$

$$\therefore\quad p_3-p_1:p_2=\lambda:1=q_3-q_1:q_2.$$

2. $p_n = b_n p_{n-1} + a_n p_{n-2}$ and $q_n = b_n q_{n-1} + a_n q_{n-2}$;

$$\therefore\ p_n q_{n-1} - q_n p_{n-1} = -a_n (p_{n-1} q_{n-2} - q_{n-1} p_{n-2}).$$

So $$p_{n-1} q_{n-2} - q_{n-1} p_{n-2} = -a_{n-1} (p_{n-2} q_{n-3} - q_{n-2} p_{n-3}),$$

$$\dots\dots\dots\dots = \dots\dots\dots\dots$$

And $$p_2 q_1 - q_2 p_1 = -a_1 a_2.$$

Hence $$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} a_1 a_2 \dots a_n.$$

3. $$\frac{63}{100} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3},$$

the successive convergents being

$$\frac{1}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{5}{8},\ \frac{12}{19},\ \frac{17}{27},\ \frac{63}{100}.$$

4. We have $$\frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{a_2} - \frac{1}{a_3} = \dots = \frac{1}{a_{n-1}} - \frac{1}{a_n};$$

$$\therefore\ \frac{a_n}{a_{n-1}} = \frac{a_{n-2}}{2a_{n-2} - a_{n-1}} = \frac{1}{2 - \dfrac{a_{n-1}}{a_{n-2}}},$$

so $$\frac{a_{n-1}}{a_{n-2}} = \frac{1}{2 - \dfrac{a_{n-2}}{a_{n-3}}}, \dots,\ a_3 = \frac{1}{2 - \dfrac{a_2}{a_1}}.$$

Hence $$\frac{a_n}{a_{n-1}} = \frac{1}{2 - \dfrac{1}{2 - \dfrac{1}{2 - \cdots - \dfrac{1}{2 - \dfrac{a_2}{a_1}}}}}.$$

5. $$\frac{1}{na_2} + \frac{1}{na_3} + \frac{1}{na_4} + \frac{1}{na_5} + \cdots$$

$$= \frac{n}{n^2 a_2} + \frac{n}{na_3} + \frac{1}{na_4} + \frac{1}{na_5} + \cdots$$

$$= \frac{n}{n^2 a_2} + \frac{n}{na_3} + \frac{n}{n^2 a_4} + \frac{n}{na_5} + \cdots + \frac{n}{na_{2r+1}} + \frac{n}{n^2 a_{2r}} + \cdots$$

$$= \frac{n}{n^2 a_2} + \frac{1}{a_3} + \frac{1}{n^2 a_4} + \frac{1}{a_5} + \cdots + \frac{1}{a_{2r+1}} + \frac{1}{n^2 a_{2r}} + \cdots$$

6. Consider the continued fractions

$$P_n = \frac{a_n}{a_n} + \frac{a_{n-1}}{a_{n-1}} + \cdots + \frac{a_1}{a_1 + 1} \text{ and } Q_n = \frac{a_n}{a_{n-1}} + \frac{a_{n-1}}{a_{n-2}} + \cdots + \frac{a_2}{a_1}.$$

Then $$P_2 = \frac{a_2}{a_2 + \dfrac{a_1}{a_1 + 1}} = \frac{1}{1 + \dfrac{1}{a_2 + Q_2}}.$$

Assume then that this relation holds good up to P_{n-1} and Q_{n-1}. Then

$$P_n = \frac{a_n}{a_n + P_{n-1}} = \frac{a_n}{a_n} + \frac{1}{1} + \frac{1}{a_{n-1} + Q_{n-1}}, \text{ by supposition}$$

$$= \frac{1}{1} + \frac{1}{a_n} + \frac{a_n}{a_{n-1} + Q_{n-1}}$$

$$= \frac{1}{1} + \frac{1}{a_n + Q_n}, \text{ since } Q_n = \frac{a_n}{a_{n-1} + Q_{n-1}}.$$

Hence the relation $P_n = \frac{1}{1} + \frac{1}{a_n + Q_n}$, or $P_n(a_n + Q_n + 1) = a_n + Q_n$ holds good for all values of n.

7. Let $$u_n = \frac{n}{n} + \frac{n-1}{n-1} + \frac{n-2}{n-2} + \cdots + \frac{2}{2} + \frac{1}{1} + \frac{1}{2}.$$

Then $$u_1 = \frac{1}{1} + \frac{1}{2} = \frac{2}{3}, \text{ and } u_2 = \frac{2}{2} + \frac{1}{1} + \frac{1}{2} = \frac{3}{4}.$$

Assume therefore that $u_n = \frac{n+1}{n+2}$; then

$$u_{n+1} = \frac{n+1}{n+1} + u_n = \frac{n+1}{n+1} + \frac{n+1}{n+2} = \frac{n+2}{n+3}.$$

Hence by induction $u_n = \frac{n+1}{n+2}$, for all values of n.

8. $$u_1 = \frac{1}{1} = \frac{2 \cdot 1}{1+1}, \quad u_2 = \frac{1}{1} - \frac{1}{4} = \frac{4}{3} = \frac{2 \cdot 2}{2+1}.$$

Now $u_{n+2} = \frac{1}{1} - \frac{1}{4} - u_n$. Hence, if $u_n = \frac{2n}{n+1}$,

$$u_{n+2} = \frac{1}{1} - \frac{1}{4} - \frac{2n}{n+1} = \frac{2(n+2)}{n+3}.$$

Hence if the theorem be true for n it will be true for $n+2$.

But the theorem is true for $n=1$, and therefore for $n=3$, and so on for all *odd* numbers.

Also the theorem is true for $n=2$, and therefore for $n=4$, and so on for all *even* numbers.

Hence $u_n = \frac{2n}{n+1}$ for all values of n.

9. It is obvious that

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} = \frac{b_1}{a_1} + \frac{b_2}{a_1 a_2};$$

and, if we assume that

$$\frac{b_2}{a_2}+\frac{b_3}{a_3}+\frac{b_4}{a_4}+\ldots=\frac{b_2}{a_2}+\frac{b_3}{a_2a_3}+\frac{b_4}{a_2a_3a_4}+\ldots,$$

we have

$$\frac{b_1}{a_1}+\frac{b_2}{a_2}+\frac{b_3}{a_3}+\ldots=\frac{b_1+\dfrac{b_2}{a_2}+\dfrac{b_3}{a_2a_3}+\ldots}{a_1}=\frac{b_1}{a_1}+\frac{b_2}{a_1a_2}+\frac{b_2}{a_1a_2a_3}+\ldots$$

Thus, if the theorem be true for any number of elements, it will be true for an additional element; hence, as it is true for two elements, it is true universally.

10. We know that

$$p_n=b_np_{n-1}+a_np_{n-2} \text{ and } p_{n+1}=b_{n+1}p_n+a_{n+1}p_{n-1}.$$

Hence $(p_n-b_np_{n-1})^2=a_n^2p^2_{n-2}$, and $(b_{n+1}p_n+a_{n+1}p_{n-1})^2=p^2_{n+1}$.

Hence $b_{n+1}a_{n+1}(p_n-b_np_{n-1})^2+b_n(b_{n+1}p_n+a_{n+1}p_{n-1})^2$

$$=b_{n+1}a_{n+1}a_n^2p^2_{n-2}+b_np^2_{n+1}.$$

The terms containing p_np_{n-1} cancel, and therefore the above is the relation required.

11. $$\frac{p_{2n+2}}{q_{2n+2}}=\frac{1}{a}+\frac{1}{b}+\frac{p_{2n}}{q_{2n}}=\frac{p_{2n}+bq_{2n}}{(ab+1)\,q_{2n}+ap_{2n}}.$$

Hence $p_{2n+2}=p_{2n}+bq_{2n}$, and $q_{2n+2}=(ab+1)\,q_{2n}+ap_{2n}$.

12. $$\frac{p_{3n+3}}{q_{3n+3}}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{p_{3n}}{q_{3n}}=\frac{(bc+1)\,q_{3n}+bp_{3n}}{(abc+a+c)\,q_{3n}+(ab+1)\,p_{3n}}.$$

Hence $$p_{3n+3}=bp_{3n}+(bc+1)\,q_{3n}.$$

13. $$p_{2n}=p_{2n-1}+bp_{2n-2} \text{ and } q_{2n}=q_{2n-1}+bq_{2n-2};$$

$$\therefore\ p_{2n}q_{2n-1}-p_{2n-1}q_{2n}=-b\,(p_{2n-1}q_{2n-2}-q_{2n-1}p_{2n-2}).$$

Also $$p_{2n-1}=p_{2n-2}+ap_{2n-3} \text{ and } q_{2n-1}=q_{2n-2}+aq_{2n-3};$$

$$\therefore\ p_{2n-1}q_{2n-2}-p_{2n-2}q_{2n-1}=-a\,(p_{2n-2}q_{2n-3}-q_{2n-2}p_{2n-3}).$$

Hence $$p_{2n}q_{2n-1}-p_{2n-1}q_{2n}=ab\,(p_{2n-2}q_{2n-3}-p_{2n-3}q_{2n-2}).$$

Similarly $$p_{2n-2}q_{2n-3}-p_{2n-3}q_{2n-2}=ab\,(p_{2n-4}q_{2n-5}-p_{2n-5}q_{2n-4}).$$

$$\ldots\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots$$

Also $$p_2q_1-p_1q_2=-ab.$$

Hence $$p_{2n}q_{2n-1}-p_{2n-1}q_{2n}=-a^nb^n.$$

14. $$\frac{p_{2n+2}}{q_{2n+2}}=\frac{1}{a}+\frac{1}{b}+\frac{p_{2n}}{q_{2n}}=\frac{p_{2n}+bq_{2n}}{ap_{2n}+(ab+1)\,q_{2n}}.$$

Hence $p_{2n+2}=p_{2n}+bq_{2n}$ and $q_{2n+2}=ap_{2n}+(ab+1)\,q_{2n}$.

But $p_{2n+2}=bp_{2n+1}+p_{2n}$ and $q_{2n+2}=bq_{2n+1}+q_{2n}$.

Hence $q_{2n}=p_{2n+1}$ and $bq_{2n+1}=ap_{2n}+abq_{2n}=ap_{2n}+abp_{2n+1}$.

15. Let $\frac{p_1}{q_1}$, ... $\frac{p_n}{q_n}$ be the convergents to $\frac{p}{q}$, so that

$$p=lp_n+p_{n-1},\ p_n=kp_{n-1}+p_{n-2},\ldots p_3=cp_2+p_1,\ p_2=ab+1,\ p_1=a.$$

Then $$\frac{p}{p_n}=l+\frac{1}{\frac{p_n}{p_{n-1}}},\quad \frac{p_n}{p_{n-1}}=k+\frac{1}{\frac{p_{n-1}}{p_{n-2}}},\quad \ldots,\quad \frac{p_3}{p_2}=c+\frac{1}{\frac{p_2}{p_1}}=c+\frac{1}{b}+\frac{1}{a}.$$

Thus $$\frac{p_n}{p}=\frac{1}{l}+\frac{1}{k}+\cdots+\frac{1}{c}+\frac{1}{b}+\frac{1}{a}.$$

Also $q=lq_n+q_{n-1},\ q_n=kq_{n-1}+q_{n-2},\ \ldots\ q_3=cq_2+q_1,\ q_2=b,\ q_1=1.$

Hence $$\frac{q}{q_n}=l+\frac{1}{\frac{q_n}{q_{n-1}}},\quad \frac{q_n}{q_{n-1}}=k+\frac{1}{\frac{q_{n-1}}{q_{n-2}}},\quad \ldots,\quad \frac{q_3}{q_2}=c+\frac{1}{\frac{q_2}{q_1}}=c+\frac{1}{b}.$$

Thus $$\frac{q_n}{q}=\frac{1}{l}+\frac{1}{k}+\cdots+\frac{1}{c}+\frac{1}{b}.$$

And, by Art. 357 I, $$\frac{p_n}{p}\sim\frac{q_n}{q}=\frac{1}{pq}.$$

16. Let $\frac{P}{Q}=a+\frac{1}{b}+\frac{1}{c}+\cdots+\frac{1}{k}+\frac{1}{l}$. Then, if $\frac{p}{q}$ be the last convergent, we have from 15,

$$\frac{Q}{q}=l+\frac{1}{k}+\cdots+\frac{1}{c}+\frac{1}{b}$$

and $$\frac{P}{p}=l+\frac{1}{k}+\cdots+\frac{1}{c}+\frac{1}{b}+\frac{1}{a}.$$

Hence, if $\frac{x}{y}$ be the convergent of $l+\frac{1}{k}+\cdots+\frac{1}{c}+\frac{1}{b}$ which just precedes $\frac{Q}{q}$, we have

$$P=aQ+x \text{ and } p=aq+y,$$

whence $$x/y=(P-aQ)/(p-aq).$$

17. From Art. 357 III, $x=\dfrac{p'+\lambda p}{q'+\lambda q}$, where λ is positive and less than 1.

Hence $pp'-qq'x^2=pp'-qq'\left(\dfrac{p'+\lambda p}{q'+\lambda q}\right)^2$

$$=\frac{pp'(q'+\lambda q)^2-qq'(p'+\lambda p)^2}{(q'+\lambda q)^2}=\frac{(p'q'-\lambda^2pq)(pq'-p'q)}{(q'+\lambda q)^2}.$$

Now $\lambda<1$, $p<p'$ and $q<q'$, therefore $p'q'-\lambda pq$ is positive. Hence $pp'-qq'x^2$ is positive or negative according as $pq'-p'q$ is positive or negative, therefore $\dfrac{pp'}{qq'}\gtrless x^2$ according as $\dfrac{p}{q}\lessgtr\dfrac{p'}{q'}$.

XXXVII. Pages 473—478.

1. (1) $$\sqrt{17}=4+\sqrt{17}-4=4+\frac{1}{\sqrt{17}+4};$$

$$\sqrt{17}+4=8+\sqrt{17}-4=8+\frac{1}{\sqrt{17}+4};\ \&c.$$

Thus $$\sqrt{17}=4+\frac{1}{8}+\frac{1}{8}+\cdots$$

(2) $$\sqrt{140}=11+\sqrt{140}-11=11+\frac{19}{\sqrt{140}+11}=11+\frac{1}{\dfrac{\sqrt{140}+11}{19}};$$

$$\frac{\sqrt{140}+11}{19}=1+\frac{\sqrt{140}-8}{19}=1+\frac{76}{19(\sqrt{140}+8)}=1+\frac{1}{\dfrac{\sqrt{140}+8}{4}};$$

$$\frac{\sqrt{140}+8}{4}=4+\frac{\sqrt{140}-8}{4}=4+\frac{76}{4(\sqrt{140}+8)}=4+\frac{1}{\dfrac{\sqrt{140}+8}{19}};$$

$$\frac{\sqrt{140}+8}{19}=1+\frac{\sqrt{140}-11}{19}=1+\frac{19}{19(\sqrt{140}+11)}=1+\frac{1}{\sqrt{140}+11};$$

$$\sqrt{140}+11=22+\sqrt{140}-11=\&c.$$

Hence $$\sqrt{140}=11+\frac{1}{1}+\frac{1}{4}+\frac{1}{1}+\frac{1}{22}+\frac{1}{1}+\frac{1}{4}+\frac{1}{1}+\frac{1}{22}+\cdots$$

(3) $$\sqrt{33}=5+\sqrt{33}-5=5+\frac{8}{\sqrt{33}+5}=5+\frac{1}{\frac{\sqrt{33}+5}{8}};$$

$$\frac{\sqrt{33}+5}{8}=1+\frac{\sqrt{33}-3}{8}=1+\frac{24}{8(\sqrt{33}+3)}=1+\frac{1}{\frac{\sqrt{33}+3}{3}};$$

$$\frac{\sqrt{33}+3}{3}=2+\frac{\sqrt{33}-3}{3}=2+\frac{24}{3(\sqrt{33}+3)}=2+\frac{1}{\frac{\sqrt{33}+3}{8}};$$

$$\frac{\sqrt{33}+3}{8}=1+\frac{\sqrt{33}-5}{8}=1+\frac{8}{8(\sqrt{33}+5)}=1+\frac{1}{\sqrt{33}+5};$$

$$\sqrt{33}+5=10+\sqrt{33}-5=\&c.$$

Hence $$\sqrt{33}=5+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{10}+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{10}+\cdots$$

(4) $$\sqrt{43}=6+\sqrt{43}-6=6+\frac{7}{\sqrt{43}+6}=6+\frac{1}{\frac{\sqrt{43}+6}{7}};$$

$$\frac{\sqrt{43}+6}{7}=1+\frac{\sqrt{43}-1}{7}=1+\frac{42}{7(\sqrt{43}+1)}=1+\frac{1}{\frac{\sqrt{43}+1}{6}};$$

$$\frac{\sqrt{43}+1}{6}=1+\frac{\sqrt{43}-5}{6}=1+\frac{18}{6(\sqrt{43}+5)}=1+\frac{1}{\frac{\sqrt{43}+5}{3}};$$

$$\frac{\sqrt{43}+5}{3}=3+\frac{\sqrt{43}-4}{3}=3+\frac{27}{3(\sqrt{43}+4)}=3+\frac{1}{\frac{\sqrt{43}+4}{9}};$$

$$\frac{\sqrt{43}+4}{9}=1+\frac{\sqrt{43}-5}{9}=1+\frac{18}{9(\sqrt{43}+5)}=1+\frac{\sqrt{43}+5}{2};$$

$$\frac{\sqrt{43}+5}{2}=5+\frac{\sqrt{43}-5}{2}=5+\frac{18}{2(\sqrt{43}+5)}=5+\frac{1}{\frac{\sqrt{43}+5}{9}};$$

$$\frac{\sqrt{43}+5}{9}=1+\frac{\sqrt{43}-4}{9}=1+\frac{27}{9(\sqrt{43}+4)}=1+\frac{1}{\frac{\sqrt{43}+4}{3}};$$

$$\frac{\sqrt{43}+4}{3}=3+\frac{\sqrt{43}-5}{3}=3+\frac{18}{3(\sqrt{43}+5)}=3+\frac{1}{\frac{\sqrt{43}+5}{6}};$$

$$\frac{\sqrt{43}+5}{6}=1+\frac{\sqrt{43}-1}{6}=1+\frac{42}{6\,(\sqrt{43}+1)}=1+\frac{1}{\frac{\sqrt{43}+1}{7}};$$

$$\frac{\sqrt{43}+1}{7}=1+\frac{\sqrt{43}-6}{7}=1+\frac{7}{7\,(\sqrt{43}+6)}=1+\frac{1}{\sqrt{43}+6};$$

$$\sqrt{43}+6=12+\sqrt{43}-6=\&c.$$

Hence $\sqrt{43}=6+\dot{\frac{1}{1}}+\frac{1}{1}+\frac{1}{3}+\frac{1}{1}+\frac{1}{5}+\frac{1}{1}+\frac{1}{3}+\frac{1}{1}+\frac{1}{1}+\dot{\frac{1}{12}}$.

(5) $\sqrt{(a^2+1)}=a+\sqrt{(a^2+1)}-a=a+\dfrac{1}{\sqrt{(a^2+1)}+a}$

$$=a+\frac{1}{a+a}+\frac{1}{a+\sqrt{(a^2+1)}}=a+\frac{1}{2a}+\frac{1}{2a}+\cdots$$

(6) $\sqrt{(a^2+2a)}=a+\sqrt{(a^2+2a)}-a=a+\dfrac{2a}{a+\sqrt{(a^2+2a)}}$

$$a+\frac{2a}{a+a}+\frac{2a}{a+\sqrt{(a^2+2a)}}=a+\frac{2a}{2a}+\frac{2a}{2a}+\cdots$$

2. $\sqrt{N}=a+\sqrt{N}-a=a+\dfrac{N-a^2}{a+\sqrt{N}}$

$$=a+\frac{b}{a+a}+\frac{b}{a+\sqrt{N}}=a+\frac{b}{2a}+\frac{b}{2a}+\cdots$$

3. (i) Let $x=1+\frac{1}{3}+\frac{1}{2}+\frac{1}{3}+\frac{1}{2}+$ ...to infinity;

then $x=1+\frac{1}{3}+\frac{1}{2}+x-1$;

$\therefore\ 3x^2-5=0$, whence $x=\sqrt{\frac{5}{3}}$.

(ii) Let $x=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+$ ...to infinity;

then $x=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{x}$; $\therefore\ 7x^2-8x-3=0$,

whence $x=\frac{1}{7}(4+\sqrt{37})$.

(iii) Let $x=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{4}+\frac{1}{5}+$ ...to infinity.

Let $y=\frac{1}{4}+\frac{1}{5}+\frac{1}{4}+\frac{1}{5}+$ ...to infinity;

then $$y=\frac{1}{4}+\frac{1}{5+y}\,;\ \therefore\ 4y^2+20y-5=0,$$

whence $$y=\frac{1}{2}(\sqrt{30}-5).$$

Hence $$x=\frac{1}{2}+\frac{1}{3+\frac{1}{2}(\sqrt{30}-5)}=\frac{\sqrt{30}+1}{2\sqrt{30}+4}=\frac{1}{52}(28-\sqrt{30}).$$

4. $$x=7+\frac{1}{14}+\frac{1}{14}+\ldots=7+\frac{1}{14+x-7}\,;\ \therefore\ x^2=50,$$

$$y=1+\frac{1}{2}+\frac{1}{2}+\ldots=1+\frac{1}{2+y-1}\,;\ \therefore\ y^2=2.$$

Hence $$x=5y.$$

5. Let $$x=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{a}+\ldots;$$

then $$x=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+x=\frac{bcd+d+b+x(bc+1)}{abcd+cd+ad+ab+1+x(abc+c+a)}.$$

The product of the roots of the quadratic for x is

$$-(bcd+b+d)/(abc+a+c).$$

But, by Art. 363, the product of the roots is

$$-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{a}+\ldots\right)\left(d+\frac{1}{c}+\frac{1}{b}+\frac{1}{a}+\ldots\right).$$

Thus

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{a}+\ldots\right)\left(d+\frac{1}{c}+\frac{1}{b}+\frac{1}{a}+\frac{1}{d}+\ldots\right)=\frac{bcd+b+d}{abc+a+c}.$$

6. We have $x=y+\dfrac{1}{y+x}$; therefore

$$x^2-y^2=1\,;\ \therefore y=x-\frac{1}{x+y}\,;$$

$$y=x-\frac{1}{x+x-}\frac{1}{x+y}=x-\frac{1}{2x-}\frac{1}{2x-}\ldots$$

7. $$x=\frac{1}{a}+\frac{1}{b+x}\,;\ \therefore ax^2+abx-b=0.$$

Change a into $2a$ and b into $2b$; then

$$2ay^2+4aby-2b=0,\ \text{and similarly}\ 3az^2+9abz-3b=0.$$

Eliminating a, ab, b from these three equations [See Art. 153, Ex. 2]; we have $$z^2(x-2y)+3z(y^2-x^2)+xy(2x-y)=0.$$

8. The continued fraction is

$$1+\frac{n^2-1^2}{3+}\ \frac{n^2-2^2}{5+}\ \frac{n^2-3^2}{7+}\ \cdots\frac{n^2-(n-2)^2}{2n-3}+\frac{n^2-(n-1)^2}{2n-1}.$$

Now $$\frac{n^2-(n-1)^2}{2n-1}=1,\quad \frac{n^2-(n-2)^2}{2n-3+1}=2,$$

$$\frac{n^2-(n-3)^2}{2n-5+2}=3,\ \ldots,\quad \frac{n^2-3^2}{7+(n-4)}=n-3,$$

$$\frac{n^2-2^2}{5+(n-3)}=n-2,\quad \frac{n^2-1}{3+(n-2)}=n-1 \text{ and } 1+n-1=n.$$

9. $$\frac{1+a^2+a^4+\ldots+a^{2n}}{a+a^3+a^5+\ldots a^{2n-1}}-\left(a+\frac{1}{a}\right)$$

$$=\frac{(1+a^2+a^4+\ldots+a^{2n})-(a^2+1)\,(1+a^2+\ldots+a^{2n-2})}{a\,(1+a^2+\ldots+a^{2n-2})}$$

$$=-\frac{a^2\,(1+a^2+\ldots+a^{2n-4})}{a\,(1+a^2+\ldots+a^{2n-2})}=-\frac{a+a^3+\ldots+a^{2n-3}}{1+a^2+\ldots+a^{2n-2}}.$$

Similarly $$\frac{1+a^2+\ldots+a^{2n-2}}{a+a^3+\ldots+a^{2n-3}}-\left(a+\frac{1}{a}\right)=-\frac{a+a^3+\ldots+a^{2n-5}}{1+a^2+\ldots+a^{2n-4}},$$

.....

$$\frac{1+a^2+a^4}{a+a^3}-\left(a+\frac{1}{a}\right)=-\frac{a}{1+a^2}=-\frac{1}{a+\frac{1}{a}}.$$

Hence $$\frac{1+a^2+a^4+\ldots+a^{2n}}{a+a^3+a^5+\ldots+a^{2n-1}}=a+\frac{1}{a}-\frac{1}{a+\frac{1}{a}-}\ \frac{1}{a+\frac{1}{a}-}\cdots$$

to n quotients.

10. We have $x=\dfrac{a}{b+y}$ and $y=\dfrac{c}{d+x}$;

$$\therefore\ xy+bx=a \text{ and } xy+dy=c;$$

$$\therefore\ bx-dy=a-c.$$

11. $x=a+\dfrac{1}{1}+\dfrac{1}{b}+\dfrac{1}{x}$, whence

$$(b+1)\,x^2-(ab+a+b-1)\,x-a-1=0.$$

Similarly $$(a+1)\,y^2-(ab+a+b-1)\,y-b-1=0.$$

Put in the first equation $\frac{1+a}{1+b}\,Y$ for x, and it becomes

$$\frac{(1+a)^2}{1+b}\,Y^2-(ab+a+b-1)\,\frac{1+a}{1+b}\,Y-(1+a)=0,$$

or
$$(1+a)\,Y^2-(ab+a+b-1)\,Y-(1+b)=0,$$

which is the second equation. Hence $\frac{x}{1+a}=\frac{y}{1+b}$.

12. Consider the continued fraction $3-\frac{2}{3}-\frac{2}{3}-\ldots$

Then, by Art. 360, the $(n-1)$th convergent is $\frac{p_{n-1}}{q_{n-1}}$ when p_{n-1} is the coefficient of x^{n-2} in the expansion of

$$\frac{3-2x}{1-3x+2x^2}=\frac{4}{1-2x}-\frac{1}{1-x},$$

and q_{n-1} is the coefficient of x^{n-2} in the expansion of

$$\frac{1}{1-3x+2x^2}=\frac{2}{1-2x}-\frac{1}{1-x}.$$

Hence
$$\frac{p_{n-1}}{q_{n-1}}=\frac{2^n-1}{2^{n-1}-1}.$$

Hence the nth convergent required

$$\frac{1}{3}-\frac{4}{\frac{2^n-1}{2^{n-1}-1}}=\frac{2^n-1}{2^n+1}.$$

Or thus:

If $n>2$, we have

$$p_n=3p_{n-1}-2p_{n-2};$$

$$\therefore\ p_n-2p_{n-1}=p_{n-1}-2p_{n-2}=p_{n-2}-2p_{n-3}=\ldots\ldots=p_3-2p_2.$$

And
$$p_3=7,\quad p_2=3.$$

Hence
$$p_n-2p_{n-1}=1.$$

So also
$$2\,.\,p_{n-1}-2^2\,.\,p_{n-2}=2,$$

$$2^2p_{n-2}-2^3p_{n-3}=2^2,\ldots,\ 2^{n-3}p_3-2^{n-2}p_2=2^{n-3}.$$

Hence by addition

$$p_n=1+2+\ldots+2^{n-3}+2^{n-2}\,.\,p_2=2^n-1,\ \text{since}\ p_2=3.$$

We find in a similar manner that $q_n=2^n+1$.

13. By Art. 360, the numerator of the nth convergent is the coefficient of x^{n-1} in $\dfrac{2+x}{1-2x-x^2}$, and the denominator is the coefficient of x^{n-1} in

$$\frac{1}{1-2x-x^2}.$$

Now $$\frac{2+x}{1-2x-x^2}=\frac{1}{2\sqrt{2}}\left\{\frac{(1+\sqrt{2})^2}{1-x(1+\sqrt{2})}-\frac{(1-\sqrt{2})^2}{1-x(1-\sqrt{2})}\right\},$$

in which the coefficient of x^{n-1} is

$$\frac{1}{2\sqrt{2}}\{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\}.$$

Also $$\frac{1}{1-2x-x^2}=\frac{1}{2\sqrt{2}}\left\{\frac{1+\sqrt{2}}{1-x(1+\sqrt{2})}-\frac{1-\sqrt{2}}{1-x(1-\sqrt{2})}\right\},$$

in which the coefficient of x^{n-1} is

$$\frac{1}{2\sqrt{2}}\{(1+\sqrt{2})^{n}-(1-\sqrt{2})^{n}\}.$$

Hence the nth convergent required is

$$\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n}-(1-\sqrt{2})^{n}}.$$

14. Since $p_n=3p_{n-1}-2p_{n-2}$, it follows that if A and B are properly chosen, p_n will be the coefficient of x^{n-1} in the expansion of $\dfrac{A+Bx}{1-3x+2x^2}$.

Now $$\frac{A+Bx}{1-3x+2x^2}=A+(B+3A)\,x+\ldots$$

Hence we must have $A=p_1=1$ and $B+3A=p_2=3$. Thus p_n is the coefficient of x^{n-1} in

$$\frac{1}{1-3x+2x^2}=\frac{2}{1-2x}-\frac{1}{1-x};$$

the coefficient of x^{n-1} is therefore 2^n-1.

Again q^n will be the coefficient of x^{n-1} in the expansion of

$$\frac{C+Dx}{1-3x+2x^2}=C+(D+3C)\,x+\ldots,$$

provided $$C=q_0=1, \text{ and } D+3C=q_1=1.$$

Thus q_n is the coefficient of x^{n-1} in $\dfrac{1-2x}{1-3x+2x^2}$,

that is in $\dfrac{1}{1-x}$; hence $q_n=1$.

The nth convergent is therefore 2^n-1.

15. p_n will be the coefficient of x^{n-1} in the expansion of

$$\frac{A+Bx}{1-(a+b)x+abx^2},$$

provided that $A=p_1=1$, and $B+(a+b)A=p_2=a+b$.

Thus p_n is the coefficient of x^{n-1} in

$$\frac{1}{(1-ax)(1-bx)}=\frac{1}{a-b}\left(\frac{a}{1-ax}-\frac{b}{1-bx}\right),$$

and is therefore $$\frac{1}{a-b}(a^n-b^n).$$

Also q_n will be the coefficient of x^{n-1} in the expansion of

$$\frac{C+Dx}{1-(a+b)x+abx^2},$$

provided that

$$C=q_1=a+b, \text{ and } D+(a+b)C=q_2=a^2+ab+b^2.$$

Thus q_n is the coefficient of x^{n-1} in

$$\frac{a+b-abx}{(1-ax)(1-bx)}=\frac{1}{a-b}\left\{\frac{a^2}{1-ax}-\frac{b^2}{1-bx}\right\};$$

and therefore $$q_n=\frac{1}{a-b}(a^{n+1}-b^{n+1}).$$

The required nth convergent is therefore $\dfrac{a^n-b^n}{a^{n+1}-b^{n+1}}$.

16.
$$p_n=(2n+1)p_{n-1}-(n^2-1)p_{n-2};$$
$$\therefore\ p_n-np_{n-1}=(n+1)\{p_{n-1}-(n-1)p_{n-2}\}.$$

So also
$$p_{n-1}-(n-1)p_{n-2}=n\{p_{n-2}-(n-2)p_{n-3}\}.$$
$$\cdots\cdots\cdots\cdots$$
$$p_3-3p_2=4\{p_2-2p_1\}.$$

Hence
$$p_n-np_{n-1}=\frac{\lfloor n+1}{\lfloor 3}(p_2-2p_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But $p_1=2$ and $p_2=10$;

$$\therefore\ p_n-np_{n-1}=\lfloor n+1, \text{ or } \frac{p_n}{\lfloor n}-\frac{p_{n-1}}{\lfloor n-1}=n+1.$$

Similarly
$$\frac{p_{n-1}}{\lfloor n-1}-\frac{p_{n-2}}{\lfloor n-2}=n,\ldots,$$

$$\frac{p_3}{\lfloor 3}-\frac{p_2}{\lfloor 2}=4,\quad \frac{p_2}{\lfloor 2}-\frac{p_1}{\lfloor 1}=3 \text{ and } p_1=2.$$

Hence $$\frac{p_n}{\lfloor n} = (n+1)+n+(n-1)+\ldots+3+2,$$

and therefore $$p_n = \frac{1}{2}n(n+3)\lfloor n.$$

The equation (i) holds good for the q's, so that

$$q_n - nq_{n-1} = \frac{\lfloor n+1}{\lfloor 3}(q_2 - 2q_1).$$

But $$q_1 = 1 \text{ and } q_2 = 2;$$

and therefore $q_n - nq_{n-1} = 0$. Hence we have

$$\frac{q_n}{\lfloor n} = \frac{q_{n-1}}{\lfloor n-1} = \ldots = \frac{q_2}{\lfloor 2} = 1.$$

The nth convergent is therefore

$$\frac{1}{2}n(n+3)\lfloor n / \lfloor n = \frac{1}{2}n(n+3).$$

17. We have

$$\frac{p_r}{q_r} = \frac{p_r}{(n^2-1)p_{r-1}+2p_r} = \frac{1}{2+(n^2-1)\dfrac{p_{r-1}}{q_{r-1}}}.$$

Now, in the limit, $\dfrac{p_r}{q_r}$ is the same as $\dfrac{p_{r-1}}{q_{r-1}}$, and hence the limiting value is given by

$$L = \frac{1}{2+(n^2-1)L}, \text{ or } n^2L^2 - (L-1)^2 = 0,$$

whence $$L = \frac{1}{1 \pm n}.$$

18. We have $$p_n = b_n p_{n-1} + a_n p_{n-2};$$

$$\therefore \frac{p_n}{p_{n-1}} = b_n + \frac{a_n}{\dfrac{p_{n-1}}{p_{n-2}}}; \quad \frac{p_{n-1}}{p_{n-2}} = b_{n-1} + \frac{a_{n-1}}{\dfrac{p_{n-2}}{p_{n-3}}}; \ldots;$$

$$\frac{p_3}{p_2} = b_3 + \frac{a_3}{\dfrac{p_2}{p_1}}; \text{ and } \frac{p_2}{p_1} = b_2.$$

Hence $$\frac{p_n}{p_{n-1}} = b_n + \frac{a_n}{b_{n-1} + \dfrac{a_{n-1}}{b_{n-2} + \ldots + \dfrac{a_3}{b_2}}}.$$

We have similarly

$$\frac{q_n}{q_{n-1}} = b_n + \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_3}{\frac{a_2}{q_1}},$$

and

$$\frac{q_2}{q_1} = b_2 + \frac{a_2}{b_1}.$$

Hence

$$\frac{q_n}{q_{n-1}} = b_n + \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_3}{b_2} + \frac{a_2}{b_1}.$$

Hence

$$\frac{p_n}{q_n} : \frac{p_{n-1}}{q_{n-1}} = b_n + \frac{a_n}{b_{n-1}} + \cdots + \frac{a_3}{b_2} : b_n + \frac{a_n}{b_{n-1}} + \cdots + \frac{a_3}{b_2} + \frac{a_2}{b_1}.$$

19. Let the fraction be $a + \frac{1}{b} + \frac{1}{c} + \cdots + \frac{1}{l} + \frac{1}{a} + \cdots$

and let $y_1, y_2, \ldots$ be $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ Let the three last convergents of the first complete period be $\frac{A}{B}, \frac{A'}{B'}$ and $\frac{A''}{B''}$, so that $A'' = p_1$ and $B'' = q_1$.

Then

$$p_1 = lA' + A \quad \text{and} \quad q_1 = lB' + B.$$

Now

$$y_n = a + \frac{1}{b} + \frac{1}{c} + \cdots + \frac{1}{l} + \frac{1}{y_{n-1}};$$

$$\therefore\ y_n = \frac{\left(l + \frac{1}{y_{n-1}}\right) A' + A}{\left(l + \frac{1}{y_{n-1}}\right) B' + B} = \frac{A'' y_{n-1} + A'}{B'' y_{n-1} + B'},$$

where $\frac{A'}{B'}$ and $\frac{A''}{B''}$ are the two last convergents of the first complete period.

20. Let $\frac{L''}{M''}, \frac{L'}{M'}, \frac{L}{M}$ be the three last convergents of

$a + \frac{1}{b} + \frac{1}{c} + \cdots + \frac{1}{k}$, $\frac{L}{M}$ being equal to P_1.

Then

$$P_r = a + \frac{1}{b} + \frac{1}{c} + \cdots + \frac{1}{k} + \frac{1}{a + P_{r-1}}$$

and

$$\sqrt{Z} = a + \frac{1}{b} + \frac{1}{c} + \cdots + \frac{1}{k} + \frac{1}{a + \sqrt{Z}}.$$

Hence

$$P_r = \frac{\left(k + \frac{1}{a + P_{r-1}}\right) L' + L''}{\left(k + \frac{1}{a + P_{r-1}}\right) M' + M''} = \frac{L\,(a + P_{r-1}) + L'}{M\,(a + P_{r-1}) + M'}.$$

Also $$\sqrt{Z}=\frac{\left(k+\frac{1}{a+\sqrt{Z}}\right)L'+L''}{\left(k+\frac{1}{a+\sqrt{Z}}\right)M'+M''}=\frac{L(a+\sqrt{Z})+L'}{M(a+\sqrt{Z})+M'}.$$

The last equation gives $La+L'-MZ-\sqrt{Z}(Ma+M'-L)=0$; and therefore, as $\sqrt{Z}$ is irrational and all the other quantities rational, we must have

$$Ma+M'-L=0 \text{ and } La+L'-MZ=0.$$

Substituting in P_r we have

$$P_r=\frac{MZ+LP_{r-1}}{L+MP_{r-1}}=\frac{Z+P_1P_{r-1}}{P_1+P_{r-1}}, \text{ since } \frac{L}{M}=P_1.$$

Hence $$\frac{P_r}{\sqrt{Z}}=\frac{Z+P_1P_{r-1}}{\sqrt{Z}\,(P_1+P_{r-1})},$$

and therefore

$$\frac{P_r+\sqrt{Z}}{P_r-\sqrt{Z}}=\left(\frac{P_1+\sqrt{Z}}{P_1-\sqrt{Z}}\right)\left(\frac{P_{r-1}+\sqrt{Z}}{P_{r-1}-\sqrt{Z}}\right)=\left(\frac{P_1+\sqrt{Z}}{P_1-\sqrt{Z}}\right)^2\left(\frac{P_{r-2}+\sqrt{Z}}{P_{r-2}-\sqrt{Z}}\right)$$
$$=\ldots=\left(\frac{P_1+\sqrt{Z}}{P_1-\sqrt{Z}}\right)^r.$$

21. Since $p_n=(a+a^{-1})p_{n-1}-p_{n-2}$, it follows as in Art. 360, that p_n is the coefficient of x^{n-1} in the expansion of

$$\frac{1}{1-(a+a^{-1})x+x^2}=\frac{1}{a-a^{-1}}\left\{\frac{a}{1-ax}-\frac{a^{-1}}{1-a^{-1}x}\right\}.$$

Hence $$p_n=\frac{1}{a-a^{-1}}(a^n-a^{-n}).$$

Also q_n is the coefficient of x^{n-1} in the expansion of

$$\frac{a+a^{-1}-x}{1-(a+a^{-1})x+x^2}=\frac{1}{a-a^{-1}}\left(\frac{a^2}{1-ax}-\frac{a^{-2}}{1-a^{-1}x}\right).$$

Hence $$q_n=\frac{1}{a-a^{-1}}(a^{n+1}-a^{-n-1}).$$

Hence $$\frac{p_n}{q_n}=\frac{a^n-a^{-n}}{a^{n+1}-a^{-n-1}}.$$

When a is greater than unity the limit of $\frac{p_n}{q_n}$ is $\frac{a^n}{a^{n+1}}=a^{-1}$; and when a is less than unity the limit of $\frac{p_n}{q_n}$ is $\frac{a^{-n}}{a^{-n-1}}=a$.

22. We know that

$$\frac{1}{a}-\frac{1}{a\,.\,b}+\frac{1}{a\,.\,b\,.\,c}-\ldots=\frac{1}{a}+\frac{a}{b-1}+\frac{b}{c-1}+\cdots \quad \text{[Art. 370, F.]}$$

Put $a=2,\ b=3=c=d=\ldots;$

then

$$\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{3}{2}+\cdots \text{ to } n \text{ quotients}$$

$$=\frac{1}{2}-\frac{1}{2\,.\,3}+\frac{1}{2\,.\,3\,.\,3}-\ldots \text{ to } n \text{ terms}$$

$$=\frac{1}{2}\left\{1-\left(-\frac{1}{3}\right)^n\right\} \Big/ \left\{1-\left(-\frac{1}{3}\right)\right\}.$$

The nth convergent can also be found as in Article 360.

23. $e^{-x}=1-x+\dfrac{x^2}{1\,.\,2}-\dfrac{x^3}{1\,.\,2\,.\,3}+\ldots$

$$=\frac{1}{1}+\frac{x}{1-x}+\frac{x}{2-x}+\frac{2x}{3-x}+\cdots+\frac{(n-1)x}{n-x}+\cdots$$

from [C.] Art. 370.

24. Put $a_1=a_2=a_3=\ldots=x$

and $b_1=a,\ b_2=b,\ b_3=c,$ &c. in [C.] Art. 370.

25. $\dfrac{1}{1}+\dfrac{1}{1}+\dfrac{2}{3}+\dfrac{4}{5}+\dfrac{6}{7}+\cdots$

$$=\frac{1}{1}-\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,4}-\frac{1}{1\,.\,2\,.\,4\,.\,6}+\ldots \quad \text{[Art. 370, F.]}$$

$$=e^{-\frac{1}{2}}.$$

26. $\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{3}{5}+\dfrac{6}{8}+\cdots+\dfrac{3(n-2)}{3n-4}+\cdots$

$$=\frac{1}{1}-\frac{1}{1\,.\,3}+\frac{1}{1\,.\,3\,.\,6}-\frac{1}{1\,.\,3\,.\,6\,.\,9}+\ldots \quad \text{[Art. 370, F.]}$$

$$=e^{-\frac{1}{3}}.$$

27. $\dfrac{1}{1\,.\,3}-\dfrac{1}{2\,.\,4}+\dfrac{1}{3\,.\,5}-\ldots+(-1)^{n-1}\dfrac{1}{n(n+2)}$

$$=\frac{1}{1\,.\,3}+\frac{1^2\,.\,3^2}{2\,.\,4-1\,.\,3}+\frac{2^2\,.\,4^2}{3\,.\,5-2\,.\,4}+\cdots+\frac{(n-1)^2(n+1)^2}{n(n+2)-(n-1)(n+1)}+\cdots$$

[Art. 370, D.]

$$=\frac{1}{3}+\frac{1^2\,.\,3^2}{5}+\frac{2^2\,.\,4^2}{7}+\cdots+\frac{(n-1)^2(n+1)^2}{2n+1}+\cdots$$

28. Assume that the formula holds good for the nth convergent; then

$$\frac{p_{n+2}}{q_{n+2}} = \frac{2}{1} - \frac{2}{9} - \frac{6.2^n - 6}{2^{n+1} - 1} = \frac{6.2^{n+2} - 6}{2^{n+3} - 1}.$$

Thus the formula will be true for the $(n+2)$th convergent if it be true for the nth; but the formula is true for the first, and therefore for all odd convergents; it is also true for the second, and therefore for all even convergents.

29. The formula is true when $n=1$, and when $n=2$; and assuming that it is true for any value of n we have

$$\frac{p_{n+2}}{q_{n+2}} = \frac{1}{3} - \frac{4}{3} - \frac{6n-1+(-1)^n}{6n+7+(-1)^n} = \frac{12n+22+2(-1)^n}{12n+38+2(-1)^n} = \frac{6(n+2)-1+(-1)^{n+2}}{6(n+2)+7+(-1)^{n+2}}.$$

Thus the formula will hold good for the $(n+2)$th convergent if it hold good for the nth; hence, as it holds good for the first and second convergents, it is true for all.

30. $$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \text{to infinity}$$

$$= \frac{1}{1} - \frac{1^2}{2+1} - \frac{2^2}{3+2} - \frac{3^3}{4+3} - \ldots \text{to infinity.}$$

But $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is infinite, and therefore

$$1 - \frac{1^2}{2+1} - \frac{2^2}{3+2} - \frac{3^2}{4+3} - \ldots = 0.$$

31. $$\sqrt{2} = (1+1)^{\frac{1}{2}} = 1 + \frac{1}{2} - \frac{1.1}{2.4} + \frac{1.1.3}{2.4.6} - \frac{1.1.3.5}{2.4.6.8} + \ldots$$

$$= 1 + \frac{1}{2} + \frac{2.1}{4-1} + \frac{4.3}{6-3} + \frac{6.5}{8-5} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1.2}{3} + \frac{3.4}{3} + \frac{5.6}{3} + \cdots$$ [Art. 370, C.]

32. $$(1+1)^{-\frac{1}{2}} = 1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \frac{1.3.5.7}{2.4.6.8} - \ldots$$

Hence $$1 - 2^{-\frac{1}{2}} = \frac{1}{2} - \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} - \frac{1.3.5.7}{2.4.6.8} + \ldots$$

$$= \frac{1}{2} + \frac{2.3}{4-3} + \frac{4.5}{6-5} + \frac{6.7}{8-7} + \cdots$$

$$= \frac{1}{2} + \frac{2.3}{1} + \frac{4.5}{1} + \frac{6.7}{1} + \cdots$$

33. From Art. 370 [C.]

$$\frac{1}{4} - \frac{3.4}{9} - \frac{5.6}{13} - \frac{7.8}{17} - \frac{9.10}{21} - \cdots$$

$$= \frac{1}{4} + \frac{1.3}{4.6} + \frac{1.3.5}{4.6.8} + \frac{1.3.5.7}{4.6.8.10} + \cdots$$

It also follows from Art. 325 that the sum of the series to infinity $= 1$.

34. $(1+x)^n = 1 + n.x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \cdots$

$$= 1 + \frac{nx}{1} - \frac{1.(n-1)x}{(n-1)x+2} - \frac{2.(n-2)x}{(n-2)x+3} - \frac{3.(n-3)x}{(n-3)x+4} - \cdots \text{[Art. 370, C.]}$$

35. Put $x = 1$, in 34; then

$$2^n = 1 + \frac{n}{1} - \frac{n-1}{n+1} - \frac{2(n-2)}{n+1} - \frac{3(n-3)}{n+1} - \cdots$$

36. It is easy to prove that

$$\left(1+\frac{x}{a}\right)\left(1+\frac{x}{a'}\right)\left(1+\frac{x}{a''}\right)\cdots = 1 + \frac{x}{a} + \frac{x(x+a)}{aa'} + \frac{x(x+a)(x+a')}{aa'a''} + \cdots$$

And, from Art. 370, the series is equal to

$$1 + \frac{x}{a} - \frac{a(x+a)}{x+a+a'} - \frac{a'(x+a')}{x+a'+a''} - \cdots$$

37. $$\frac{1}{n} - \frac{n}{2n+1} - \frac{2n}{3n+1} - \cdots$$

$$= \frac{1}{n} + \frac{1}{n.2n} + \frac{1}{n.2n.3n} + \cdots$$

$$= e^{\frac{1}{n}} - 1.$$

Also $e^{-\frac{1}{n}} = 1 - \frac{1}{n} + \frac{1}{n.2n} - \frac{1}{n.2n.3n} + \cdots$

$$= 1 - \frac{1}{n} + \frac{n}{2n-1} + \frac{2n}{3n-1} + \cdots;$$

$$\therefore \frac{1}{1-e^{-\frac{1}{n}}} = n + \frac{n}{2n-1} + \frac{2n}{3n-1} + \cdots;$$

$$\therefore n - 1 + \frac{n}{2n-1} + \frac{2n}{3n-1} + \cdots = \frac{1}{1-e^{-\frac{1}{n}}} - 1 = \frac{1}{e^{\frac{1}{n}}-1};$$

$$\therefore \frac{1}{n-1} + \frac{n}{2n-1} + \frac{2n}{3n-1} + \cdots = e^{\frac{1}{n}} - 1.$$

Art. 377. Ex. 2. $I\left(\frac{80}{3}\right)+I\left(\frac{80}{3^2}\right)+I\left(\frac{80}{3^3}\right)=26+8+2=36.$

$$I\left(\frac{80}{5}\right)+I\left(\frac{80}{5^2}\right)=16+3=19.$$

Ex. 3. $I\left(\frac{1000}{7}\right)+I\left(\frac{1000}{7^2}\right)+I\left(\frac{1000}{7^3}\right)=142+20+2=164.$

Art. 379. Ex. 1.

$$n(n+1)(2n+1)=n(n+1)(n+2)+(n-1)n(n+1).$$

Ex. 2. Let $n=2m+1$; then $(n^2+3)(n^2+7)$

$=(4m^2+4m+4)(4m^2+4m+8)=16(m^2+m+1)(m^2+m+2)=m(32).$

Ex. 3. Let $n=2m+1$; then

$$n^4+4n^2+11=16m^4+32m^3+40m^2+24m+16$$
$$=16(m^4+2m^3+2m^2+m+1)+8m(m+1)=M(16).$$

Ex. 4. $7^{2n+1}+1=(8-1)^{2n+1}+1=\{8^{2n+1}-(2n+1)8^{2n}+\ldots-1\}+1=M(8).$

Ex. 5. $19^{2n}-1=(20-1)^{2n}-1=(20^{2n}-2n\,.\,20^{2n-1}$
$$+\ldots-2n\,.\,20+1)-1=M(40).$$

Also $19^{2n}-1=(18+1)^{2n}-1=(18^{2n}+2n\,.\,18^{2n-1}+\ldots+1)-1=m(9).$

Hence $19^{2n}-1=M(40\times 9)$, since 40 and 9 are prime to each other.

Ex. 6. $n(n^2-1)(n^2-4)=(n-2)(n-1)n(n+1)(n+2).$

Of the five consecutive numbers $n-2$, $n-1,\ldots n+2$, one must be a multiple of 5; also, since n is not a multiple of 3, two of the others must be multiples of 3, and therefore the continued product must be a multiple of 9; and, since n is odd, $n-1$ and $n+1$ are both even, one of the two being divisible by 4, and therefore their product by 8. Thus $n(n^2-1)(n^2-2)$ is a multiple of 5, of 9 and of 8, and therefore of $5\times 9\times 8$.

Art. 380. Ex. 1. By Fermat's Theorem

$$2^{n-1}=1+M(n),\ 3^{n-1}=1+M(n),\ \ldots,\ (n-1)^{n-1}=1+M(n).$$

Hence $$1^{n-1}+2^{n-1}+\ldots+(n-1)^{n-1}=n-1+M(n).$$

Ex. 2. By Fermat's Theorem

$$a^{n-1}=1+M(n),\text{ and } b^{n-1}=1+M(n);\ \therefore\ a^{n-1}-b^{n-1}=M(n).$$

Ex. 3. By Fermat's Theorem $n(n^4-1)=M(5)$. Also $n(n^4-1)=M(6)$.

Ex. 4. $n(n^6-1)=M(7)$, by Fermat's Theorem.

Also $n(n^6-1)=(n-1)\,n(n+1)(n^4+n^2+1)=M(6)$.

Hence $n(n^6-1)=M(7\times 6)$.

Ex. 5. x^6-1, x^4-1 and x^2-1 are all factors of $x^{12}-1$; and, by Fermat's Theorem, $x^6-1=M(7)$, $x^4-1=M(5)$, $x^2-1=M(3)$ and $x^{12}-1=M(13)$. Hence, as 13, 7, 5 and 3 are all prime to one another,

$$x^{12}-1=M(1365).$$

So also $$y^{12}-1=M(1365),$$

and therefore $$x^{12}-y^{12}=M(1365).$$

Ex. 6. $m^{n-1}-1=M(n)$, and $n^{m-1}-1=M(m)$;

$$\therefore\ m^{n-1}n^{m-1}-m^{n-1}-n^{m-1}+1=M(n)\,.\,M(m)=M(mn);$$

$$\therefore\ m^{n-1}+n^{m-1}-1=M(mn).$$

Ex. 7. $(np)^{m-1}-1=M(m)$, $(pm)^{n-1}-1=M(n)$, and $(mn)^{p-1}-1=M(p)$.

Hence, by multiplication,

$$M(mnp)+(np)^{m-1}+(pm)^{n-1}+(mn)^{p-1}-1=M(m)\,M(n)\,M(p)=M(mnp).$$

Ex. 8. By Fermat's Theorem $n(n^4-1)=M(5)$;

$$\therefore\ n=M(5),\ \text{or else}\ n^4=1+M(5).$$

Ex. 9. By Fermat's Theorem $n(n^{12}-1)=M(13)$;

$$\therefore\ n=M(13),\ \text{or else}\ n^{12}=1+M(13).$$

Ex. 10. By Fermat's Theorem $n(n^{16}-1)=M(17)$;

$$\therefore\ n=M(17),\ \text{or else}\ (n^8-1)(n^8+1)=M(17),$$

and therefore $$n^8\pm 1=M(17).$$

Art. 381. Ex. 3. $1000=2^3\cdot5^3$, $3600=2^4\cdot3^2\cdot5^2$, and $14553=3^3\cdot7^2\cdot11$. Hence the numbers required are

$(3+1)(3+1)$, $(4+1)(2+1)(2+1)$ and $(3+1)(2+1)(1+1)$ respectively.

Ex. 4. $6=2\,.\,3$; $\therefore$ sum of its divisors $=(1+2)(1+3)-6=6$.

$28=2^2\,.\,7$; $\therefore$ sum of its divisors $=(1+2+2^2)(1+7)-28=28$.

$496=2^4\,.\,31$; $\therefore$ sum of its divisors $=(1+2+2^2+2^3+2^4)(1+31)-496=496$.

Ex. 5. A number with six divisors must be of the form x^2y, where x and y are primes, and the smallest number is $2^2\,.\,3=12$.

Ex. 6. A number with fifteen divisors must be of the form x^2y^4, and the smallest number is $3^2\,.\,2^4$.

Ex. 7. A number with twenty divisors must be of the form x^4y^3 or of the form x^4yz; the smallest numbers of these forms are $2^4\,.\,3^3$ and $2^4\,.\,3\,.\,5$.

Ex. 8. $4725 = 3^3 \cdot 5^2 \cdot 7$. Hence the numbers required are (i) $3 \cdot 7$ and (ii) $5 \cdot 7^2$.

Art. 384. Ex. 4. Every number is of the form $7m$, $7m \pm 1$, $7m \pm 2$ or $7m \pm 3$. Hence every cube is of the form $7m$, $7m \pm 1$, $7m \pm 1$ or $7m \mp 1$.

Again, every number is of the form $9m$, $9m \pm 1$, $9m \pm 2$, $9m \pm 3$ or $9m \pm 4$. Hence every cube is of the form $9m$, $9m \pm 1$, $9m \mp 1$, $9m$ or $9m \pm 1$.

Ex. 5. Every number is of the form $5m$, $5m \pm 1$ or $5m \pm 2$. Hence every fourth power is of the form $5m$ or $5m + 1$.

Ex. 6. The last digit in $(10x + y)^2$ is the same as the last digit in y^2; and y^2 is 0, 1, 4, 9, 16, 25, 36, 49, 64 or 81. Hence the last digit of any square must be 0, 1, 4, 5, 6 or 9.

Ex. 7. $(10x + y)^2 = 100x^2 + 20xy + y^2$, and y^2 is 1, 9, 25, 49 or 81. Hence, as $20xy$ is an even multiple of 10, the last figure but one must be even.

Ex. 8. $y^{4n+1} - y = y\,(y^4 - 1) \times$ an integral expression.

By Fermat's Theorem $y\,(y^4 - 1) = M\,(5)$, and

$$y\,(y^4 - 1) = y\,(y - 1)\,(y^3 + y^2 + y + 1) = M\,(2).$$

Hence $y\,(y^4 - 1)$, and therefore also $y^{4n+1} - y$, is a multiple of 10. Hence y^{4n+1} and y must end in the same digit.

Ex. 9. $n\,(n+1)\,(n+2)\,(n+3) = (n^2 + 3n + 1)^2 - 1$, and two squares cannot differ by unity.

XXXVIII. Pages 490—491.

1. Let x and y be the numbers. Then $x^2 - 1 = M\,(3)$ by Fermat's Theorem; also, since x is odd, $x^2 = M(8) + 1$. Hence $x^2 = M(24) + 1$; so also $y^2 = M(24) + 1$, and therefore $x^2 \sim y^2 = M(24)$.

2. $n\,(n^2 - 1)\,(n^2 - 4)\,(n^2 - 9) = (n - 3)\,(n - 2)\,(n - 1)\,n\,(n + 1)\,(n + 2)\,(n + 3)$.

The seven consecutive integers from $n - 3$ to $n + 3$ must contain one multiple of 7, one of 5, and two of 3; also, since n is odd, $n - 3$, $n - 1$, $n + 1$ and $n + 3$ are four consecutive even numbers, and therefore one of the four is divisible by 8 and another by 4, so that their continued product is divisible by 2^7.

Hence $$n\,(n^2 - 1^2)\,(n^2 - 2^2)\,(n^2 - 3^2) = M\,(2^7 \cdot 3^2 \cdot 5 \cdot 7).$$

3. Since n is odd $(n + 2m)^{n-1} - 1$ is divisible by $(n + 2m)^2 - 1$.

Hence $(n + 2m)^n - (n + 2m)$ is divisible by $(n + 2m - 1)\,(n + 2m)\,(n + 2m + 1)$. Since n is odd, two of the three consecutive integers are even and therefore the product is divisible by 8; also one of the three must be divisible by 3. Hence $(n + 2m - 1)\,(n + 2m)\,(n + 2m + 1)$ is a multiple of 24.

4. $a^{4m+p} - a^{4n+p} = a^{4n+p}(a^{4l} - 1)$, where $l = m - n$.

Now $a\,(a^{4l} - 1) = M(5)$ by Fermat; also $a\,(a^2 - 1) = (a - 1)\,a\,(a + 1) = M(6)$. Hence $a\,(a^{4l} - 1) = M(30)$.

5. $N - xy = N - (N - a^2)\,\{(a+1)^2 - N\} = N^2 - 2a\,(a+1)\,N + a^2\,(a+1)^2$
$= \{N - a\,(a+1)\}^2$.

6. The number of integers which are less than 990 and prime to it is

$$990\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 264.$$

Also between 990 and 1000 there are two numbers, namely 991 and 997 which are prime to 2, 3 and 5. Hence there are 266 numbers less than 1000 and prime to 2, 3 and 5.

7. Since $\frac{P}{p} + \frac{Q}{q} + \frac{R}{r}$ is an integer, $\frac{Pqr}{p} + Qr + Rq$ is an integer, and therefore Pqr is divisible by p. But q and r are both prime to p; therefore P must be divisible by p. Similarly $\frac{Q}{q}$ and $\frac{R}{r}$ are integers.

8. $284 = 2^2 \,.\, 71$, and the sum of all the divisors of 284

$$= (1 + 2 + 2^2)\,(1 + 71) - 284 = 220.$$

$$220 = 2^2 \,.\, 5 \,.\, 11,$$

and the sum of all the divisors of 220

$$= (1 + 2 + 2^2)\,(1 + 5)\,(1 + 11) - 220 = 284.$$

9. The sum of the divisors of $2^{n-1}(2^n - 1)$, if $2^n - 1$ is prime, is

$$(1 + 2 + \ldots + 2^{n-1})\,(1 + 2^n - 1) - 2^{n-1}\,(2^n - 1) = 2^{n-1}\,(2^n - 1).$$

10. By Fermat's Theorem $x^{16} - 1 = M(17)$, if x is prime to 17; and $x^{16} - 1 = M(x^4 - 1) = M(5)$, if x is prime to 5. And, if x is odd, $x^2 - 1 = M(8)$. Hence $x^{16} - 1 = M(17 \times 5 \times 8)$, if x be prime to 2, 5 and 17. The numbers required are therefore 3, 7, 9, 11, 13, and 19.

11. Let N be any number, then N is of one of the forms $9m$, $9m \pm 1$, $9m \pm 2$, $9m \pm 3$, or $9m \pm 4$. Hence N^2 is of one of the forms $9m$, $9m + 1$, $9m + 4$ or $9m + 7$; and N^3 is of one of the forms $9m$, $9m \pm 1$. Now a number is equal to a multiple of 9 plus the sum of its digits; and from the above it follows that the sum of the digits of a perfect square, or of a perfect cube, can never be 15.

12. $$(2n)^2 = (n^2 + 1)^2 - (n^2 - 1)^2;$$

and $$(2n+1)^2 = \left\{\frac{(2n+1)^2 + 1}{2}\right\}^2 - \left\{\frac{(2n+1) - 1}{2}\right\}^2.$$

13. The number is $M(7)-6$, $M(8)-6$ and $M(9)-6$. Now all numbers which are multiples of 7, 8 and 9 are multiples of 504. Hence all the numbers which satisfy the conditions are included in the formula $504m-6$, the least number being 498.

14. By Fermat's Theorem $N^{n-1}=1+M(n)$;

$$\therefore N^{(n-1)n}=\{1+M(n)\}^n=1+nM(n)+\ldots=1+M(n^2).$$

$$N^{(n-1)n^2}=\{N^{(n-1)n}\}^n=\{1+M(n^2)\}^n=1+nM(n^2)+\ldots=1+M(n^3);$$

and so on, so that $N^{(n-1)n^{r-1}}=1+M(n^r)$.

15. By the preceding question $N^{(n-1)n}-1=M(n^2)$, that is

$$(N^{\frac{1}{2}n(n-1)}+1)(N^{\frac{1}{2}n(n-1)}-1)=M(n^2).$$

Hence $$N^{\frac{1}{2}n(n-1)}\pm 1=M(n^2).$$

16.
$$a_r-(-1)^r(r+1)=\frac{(p-2)(p-3)\ldots(p-r-1)}{\lfloor r}-(-1)^r(r+1)$$
$$=\{(p-2)(p-3)\ldots(p-\overline{r+1})-(-1)^r\lfloor r+1\}/\lfloor r$$
$$=\{M(p)+(-1)^r\lfloor r+1-(-1)^r\lfloor r+1\}/\lfloor r$$
$$=M(p)/\lfloor r=M(p),$$

since p is prime and r less than p.

17. Let the primes be x, $x+a$ and $x+2a$; then, by Ex. 1, if $x>3$, $(x+a)^2-x^2=M(24)$, and $(x+2a)^2-(x+a)^2=M(24)$. Hence $2ax+a^2=M(24)$, and $2ax+3a^2=M(24)$; therefore $2a^2=M(24)$. Hence $a^2=M(12)$, from which it is obvious that $a=M(6)$.

18. Let n be any prime factor of the numerator and of the denominator; then we have to shew that n occurs to at least as high a power in the numerator as in the denominator; this is proved as in Art. 378 provided

$$I\left(\frac{2a}{k}\right)+I\left(\frac{2b}{k}\right)\nless I\left(\frac{a}{k}\right)+I\left(\frac{b}{k}\right)+I\left(\frac{a+b}{k}\right),$$

for all values of k.

Let $a=mk+\alpha$ and $b=m'k+\beta$, where each of the quantities α, β is zero or a positive integer less than k.

Then we have to shew that

$$I\left(\frac{2\alpha}{k}\right)+I\left(\frac{2\beta}{k}\right)\nless I\left(\frac{\alpha+\beta}{k}\right).$$

Now $I\left(\frac{\alpha+\beta}{k}\right)$ cannot be greater than 1, and cannot be equal to 1 unless either α or β is greater than $\frac{k}{2}$ in which case $I\left(\frac{2\alpha}{k}\right)+I\left(\frac{2\beta}{k}\right)\nless 1$.

19. As in 18, it is sufficient to prove that $I\left(\frac{2n}{k}\right) \nless I\left(\frac{n+1}{k}\right) + I\left(\frac{n}{k}\right)$, for all values of k.

Let $n = mk + a$; then we have to prove that

$$2m + I\left(\frac{2a}{k}\right) \nless 2m + I\left(\frac{a+1}{k}\right),$$

which is obvious.

20. By Art. 378

$$\frac{(nr-1)!}{(nr-r)!\,(r-1)!}, \text{ and therefore } \frac{(nr)!}{\{(n-1)r\}!\,r!\,n} \text{ is an integer.}$$

Similarly $\frac{\{(n-1)r\}!}{\{(n-2)r\}!\,r!\,(n-1)}$, $\frac{\{(n-2)r\}!}{\{(n-3)r\}!\,r!\,(n-2)}$, ...

$$\ldots \frac{(3r)!}{(2r)!\,r!\,3} \text{ and } \frac{(2r)!}{r!\,r!\,2}$$

are all integers.

Hence by multiplication $\frac{(nr)!}{n!\,(r!)^n}$ is an integer.

21. We have identically

$$(a_1^2 + a_2^2 + \ldots + a_n^2)\,(b_1^2 + b_2^2 + \ldots + b_n^2) = (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2 + \Sigma(a_r b_s - a_s b_r)^2,$$

where the number of terms in Σ is the number of combinations of n things two together. Thus the product is expressed as the sum of $1 + \frac{1}{2}n(n-1)$ squares.

22. Any number is of the form $3m$, or $3m \pm 1$; and therefore every square is of the form $3m$ or $3m+1$. Hence $a^2 + b^2$ cannot be divisible by 3 unless a^2 and b^2 are both of the form $3m$, in which a and b must both be multiples of 3.

Any number is of the form $7m$, $7m \pm 1$, $7m \pm 2$ or $7m \pm 3$; and therefore every square is of the form $7m$, $7m+1$, $7m+4$ or $7m+2$. Hence $a^2 + b^2$ cannot be of the form $7m$ unless both are of the form $7m$, in which case a and b are both multiples of 7.

Similarly for 11, since every square is of the form $11m$, $11m+1$, $11m+4$, $11m+9$, $11m+5$ or $11m+3$.

23. By Art. 384, Ex. 3, $ab = M(12)$. Also a^2 and b^2 are of the form $7m$, $7m+1$, $7m+2$ or $7m+4$. Hence, as $a^2 + b^2$ is a square, unless a or b is a multiple of 7, it follows that a^2 and b^2 are *both* of the form $7m+1$, or *both* of the form $7m+2$, or *both* of the form $7m+4$. Thus either ab or $a^2 - b^2$ is a multiple of 7.

24. If $a^2+b^2=3(c^2+d^2)$, a^2+b^2 must be a multiple of 3, and therefore, by 22, a and b must both be multiples of 3. Let $a=3a_1$ and $b=3b_1$. Then we have $c^2+d^2=3(a_1^2+b_1^2)$. Hence c and d must be multiples of 3; suppose them equal to $3c_1$ and $3d_1$ respectively, then we have

$$a_1^2+b_1^2=3(c_1^2+d_1^2).$$

Repeating the process, we see that a, b, c, d must all be divisible an indefinite number of times by 3, which is impossible.

The same reasoning applies to 7 and 11.

25. a^2 is of the form $3m$ or $3m+1$, $2b^2$ of the form $3m$ or $3m+2$, and c^2 of the form $3m$ or $3m+1$.

Hence a^2 and c^2 must both be of the form $3m$ or both of the form $3m+1$; and therefore $a^2-b^2=M(3)$.

Again a^2 is of the form $16m$, $16m+4$, or $8m+1$; $2b^2$ is of the form $16m$, $16m+8$, or $16m+2$; and c^2 is of the form $16m$, $16m+4$ or $8m+1$. Hence a^2 and c^2 must *both* be of the form $16m$, or *both* of the form $16m+4$, or *both* of the form $8m+1$; and therefore $a^2-b^2=M(8)$.

XXXIX. Pages 501—502.

1. (i) Let $$u_n=2^{2n+1}-9n^2+3n-2;$$

then $$u_{n+1}=2^{2n+3}-9(n+1)^2+3(n+1)-2.$$

Hence $$u_{n+1}-4u_n=27n^2-27n=27n(n-1)=M(54).$$

Hence u_{n+1} will be a multiple of 54, if u_n be a multiple of 54. But

$$u_1=0;\ \therefore u_2=M(54);\ \therefore u_3=M(54),\ \&c.$$

(ii) $$5^{2n+1}+n^5-5n^3+4n-5=5(1+24)^n+n^5-5n^3+4n-5$$
$$=5\left\{1+n\,.\,24+\frac{n(n-1)}{1\,.\,2}24^2+\ldots\right\}+n^5-5n^3+4n-5$$
$$=M(120)+n^5-5n^3+4n=M(120)+(n-2)(n-1)n(n+1)(n+2)$$
$$=M(120).$$

(iii) $$4^{2n+1}+3^{n+2}=4(13+3)^n+9\,.\,3^n=4\{3^n+M(13)\}+9\,.\,3^n$$
$$=13\,.\,3^n+M(13)=M(13).$$

(iv) $$3^{4n+2}+2\,.\,4^{3n+1}=9\,.\,81^n+8\,.\,64^n=9(85-4)^n+8(68-4)^n$$
$$=M(85)+9(-4)^n+M(68)+8(-4)^n=M(17)+17(-4)^n=M(17).$$

2. If possible, let r^2b^2 and s^2b^2 give the same remainder. Then

$$r^2b^2-s^2b^2=M(a),$$

that is $(r+s)(r-s)b^2=M(a)$. But this is impossible since a is a prime, and b is prime to a, and $r+s$ and $r-s$ are both less than a and therefore both prime to a.

3. $\lfloor\underline{4n+1} = M(4n+1)$, since $4n+1$ is a prime.

Hence

$$[\{(4n+1)-1\}\{(4n+1)-2\}\ldots\{(4n+1)-2n\}]\,\lfloor\underline{2n}+1 = M(4n+1);$$

$$\therefore\ M(4n+1)+(-1)^{2n}\{\lfloor\underline{2n}\}^2+1 = M(4n+1);$$

$$\therefore\ \{\lfloor\underline{2n}\}^2+1 = M(4n+1).$$

Again, if $4n-1$ is a prime,

$$\lfloor\underline{4n-2}+1 = M(4n-1);$$

$$\therefore\ [\{(4n-1)-1\}\{(4n-1)-2\}\ldots\{(4n-1)-(2n-1)\}]\,\lfloor\underline{2n-1}+1 = M(4n-1);$$

$$\therefore\ M(4n-1)+(-1)^{2n-1}\,\lfloor\underline{2n-1}\,\lfloor\underline{2n-1}+1 = M(4n-1);$$

$$\therefore\ \{\lfloor\underline{2n-1}\}^2-1 = M(4n-1).$$

4. Since n is a prime number, $\lfloor\underline{n-1}+1 = M(n)$.

But
$$\lfloor\underline{n-1} = \lfloor\underline{n-r}\,(n-r+1)(n-r+2)\ldots(n-1)$$
$$= M(n)+(-1)^{r-1}\,\lfloor\underline{r-1}\,\lfloor\underline{n-r}.$$

Hence
$$1+(-1)^{r-1}\,\lfloor\underline{r-1}\,\lfloor\underline{n-r} = M(n).$$

5. Let $2r+1$ be a *prime* factor of m^2+n^2.

Then m and n are both prime to $2r+1$.

Hence $m^{2r} = 1+M(2r+1)$; and $m^2+n^2 = M(2r+1)$;

$$\therefore\ \{M(2r+1)-n^2\}^r = 1+M(2r+1);$$

$$\therefore\ (-1)^r n^{2r} = 1+M(2r+1);$$

but
$$n^{2r} = 1+M(2r+1);$$

$$\therefore\ (-1)^r\{1+M(2r+1)\} = 1+M(2r+1),$$

whence it follows that r is *even*.

Hence every odd *prime* divisor of m^2+n^2 is of the form $4k+1$, and the product of any number of quantities of the form $4k+1$ is obviously of the same form.

6. Let
$$S = \frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots;$$

then, if $n>1$, we have

$$S\left(1-\frac{1}{2^n}\right) = \frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\ldots,$$

the multiples of 2 being all absent.

Similarly
$$S\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right) = \frac{1}{1^n}+\frac{1}{5^n}+\frac{1}{7^n}+\ldots,$$

the multiples of 2 and 3 being all absent.

Hence
$$S\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\ldots = 1,$$

2, 3, 5,... being all the prime numbers.

7. If a is prime to n, $n-a$ is also prime to n. Hence the numbers occur in pairs such that the arithmetic mean of each pair is $\frac{n}{2}$; therefore the arithmetic mean of the whole is $\frac{n}{2}$.

8. The number of numbers less than N and prime to it is

$$N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots;$$

and by the preceding example their arithmetic mean is $\frac{1}{2}N$; hence their sum is

$$\frac{1}{2}N^2\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots$$

Let S be the sum of the squares of the numbers which are less than N and *not* prime to N; then, as in Art. 380,

$$S=\Sigma\left\{a^2+2^2a^2+\ldots\left(\frac{N}{a}\right)^2a^2\right\}-\Sigma\left\{a^2b^2+2^2a^2b^2+\ldots+\left(\frac{N}{ab}\right)^2a^2b^2\right\}$$
$$+\Sigma\left\{a^2b^2c^2+2^2a^2b^2c^2+\ldots\left(\frac{N}{abc}\right)^2a^2b^2c^2\right\}-\ldots$$

Now $$a^2+2^2a^2+\ldots+\left(\frac{N}{a}\right)^2a^2=\frac{1}{6}a^2\frac{N}{a}\left(\frac{N}{a}+1\right)\left(\frac{2N}{a}+1\right)$$
$$=\frac{1}{6}\frac{N}{a}(N+a)(2N+a)=\frac{1}{6}\left\{\frac{2N^3}{a}+3N^2+Na\right\}.$$

Hence if there are r of the letters a, b, c, ..., we have

$$6S=2N^3\left\{\Sigma\frac{1}{a}-\Sigma\frac{1}{ab}+\Sigma\frac{1}{abc}-\ldots\right\}$$
$$+3N^2\left\{r-\frac{r(r-1)}{1\cdot2}+\frac{r(r-1)(r-2)}{1\cdot2\cdot3}-\ldots\right\}$$
$$+N\{\Sigma a-\Sigma ab+\Sigma abc-\ldots\}$$
$$=2N^3\left\{1-\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\right\}$$
$$+3N^2+N\{1-(1-a)(1-b)(1-c)\ldots\}.$$

Now the sum of the squares of the numbers from 1 to N inclusive is

$$\frac{1}{6}N(N+1)(2N+1),$$

hence the sum of the squares of all the numbers less than N and prime to N is

$$\frac{1}{3}N^3\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots+\frac{1}{6}N(1-a)(1-b)(1-c)\ldots$$

9. Let $n = a^\alpha b^\beta c^\gamma \ldots$, a, b, c, ... being primes; then any divisor, d, may be represented by $a^p b^q c^r \ldots$, where p, q, r, ... are not greater than α, β, γ, ... respectively.

We know that

$$\phi(d) = d\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots = a^{p-1}(a-1)\,b^{q-1}(b-1)\,c^{r-1}(c-1)\ldots$$

Now all the different values of $\phi(d)$, for all possible values of p, q, r, ..., are the different terms of the continued product of the following expressions,

$$1+(a-1)+a(a-1)+a^2(a-1)+\ldots+a^{\alpha-1}(a-1),$$
$$1+(b-1)+b(b-1)+b^2(b-1)+\ldots+b^{\beta-1}(b-1),$$
$$1+(c-1)+c(c-1)+c^2(c-1)+\ldots+c^{\gamma-1}(c-1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence $\Sigma\phi(d)$ is equal to the continued product of the expressions, which are respectively a^α, b^β, c^γ, ...

Thus $$\Sigma\phi(d) = n.$$

10. If $\frac{a}{b}$ be reduced to a circulating decimal with $2p$ figures in its circulating period, we have

$$\frac{a}{b} = \frac{N}{10^{2p}-1},$$

where N is an integer.

Hence $a(10^{2p}-1) = M(b)$; and therefore, as a is prime to b, $10^{2p}-1 = M(b)$; and $2p$ *is the lowest power of* 10 *for which this is true*, for otherwise $\frac{a}{b}$ could be expressed as a circulating decimal with fewer than $2p$ figures in its period.

Then, since $10^{2p}-1 = (10^p-1)(10^p+1) = M(b)$ and $10^p-1 \neq M(b)$, and b is prime, it follows that $10^p+1 = M(b)$.

Now let $$\frac{a}{b} = .\dot{c}_1 c_2 c_3 \ldots c_p c_{p+1} c_{p+2} \ldots \dot{c}_{2p};$$

then $$10^p \times \frac{a}{b} = c_1 c_2 \ldots c_p . \dot{c}_{p+1} c_{p+2} \ldots c_{2p} c_1 \ldots \dot{c}_p.$$

Hence, as $$10^p+1 = M(b),$$

it follows that $.\dot{c}_1 c_2 \ldots c_p c_{p+1} \ldots \dot{c}_{2p} + .\dot{c}_{p+1} c_{p+2} \ldots c_{2p} c_1 \ldots \dot{c}_p = \text{an integer} = 1,$

from which it follows that

$$c_1 + c_{p+1} = c_2 + c_{p+2} = c_3 + c_{p+3} = \ldots = c_p + c_{2p} = 9.$$

11. If $$\frac{1}{p} = .\dot{a}_1 a_2 a_3 \ldots \dot{a}_{p-1},$$

we have $10 = a_1 p + r_1$, $10 r_1 = a_2 p + r_2$, $10 r_2 = a_3 p + r_3$, &c.

Now p is prime to 10, for otherwise $\frac{1}{p}$ would not be reducible to a pure circulator; hence p is prime to r_1, r_2, r_3,.... But, as r_1, r_2, r_3... must be all different, they are the numbers 1, 2, 3,..., not of course in order. Hence p must be prime to 1, 2,..., $p-1$, that is must be a prime number.

Now, since r_1, r_2, ... include all the numbers 1, 2, 3,..., $p-1$, it follows that when $\frac{k}{p}$ is reduced to a decimal, the recurring digits will be the same as before, beginning with that one for which the preceding remainder was k.

12. Since $\frac{1}{P}$ has a circulating period of p figures, $10^p - 1 = M(P)$. So also $10^q - 1 = M(Q)$, $10^r - 1 = M(R)$, &c. Now, if n be the least common multiple of p, q, r..., $10^n - 1$ is a multiple of $10^p - 1$, $10^q - 1$, $10^r - 1$, &c.; and n is the least number for which this is true.

Hence $10^n - 1$ is a multiple of P, Q, R,...; and therefore, as P, Q, R,... are primes, $10^n - 1 = M(PQR\ldots)$, and this is not true for any power of 10 lower than 10^n. This shews that $\frac{1}{PQR\ldots}$ has a circulating period of n figures.

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1. (1) The penultimate convergent to $\frac{7}{15}$ is $\frac{1}{2}$. Then

$$7.2+15(-1)=-1;\ \therefore\ 7(-2.59)+15(59)=59.$$

Hence the general solution in integers is $x=-118+15m$, $y=59-7m$, and the only solution in positive integers is $x=2$, $y=3$.

(2) The penultimate convergent to $\frac{8}{13}$ is $\frac{3}{5}$. Then

$$8.5-13.3=1;\ \therefore\ 8(5.138)+13(-3.138)=138.$$

Hence $x=690-13m$, and $y=-414+8m$ is the general solution in integers; and if x and y are both positive $m\nmid 53$ and $m\nmid 52$.

(3) The penultimate convergent to $\frac{7}{9}$ is $\frac{3}{4}$. Then

$$7.4-9.3=1;\ \therefore\ 7(4.100)+9(-3.100)=100.$$

Hence the general solution in integers is $x=400-9m$, $y=-300+7m$; and, if x and y are both positive, $m\nmid 44$ and $m\nmid 43$.

(4) The penultimate convergent to $\frac{15}{71}$ is $\frac{4}{19}$. Then

$$15.19-71.4=1;\ \therefore\ 15(19.10653)+71(-4.10653)=10653.$$

Hence $x=19.10653-71m$, $y=-4.10653+15m$; and x and y are both positive if $m\nmid 2850$, and $m\nmid 2841$.

2. (1) Since $2(-1)+3.1=1$, we have $2(-133)+3(133)=133$; $\therefore$ the general solution in positive integers is given by $x=-133+3m$, $y=133-2m$, m having any value from 45 to 66 inclusive.

(2) We have $7(-3)+11(2)=1$; therefore

$$7(-3.2312)+11(2.2312)=2312.$$

Hence the general solution in positive integers is given by

$$x=-6936+11m,\ y=4624-7m,$$

m having any value from 631 to 660 inclusive.

3. (1) We have

$$7.2-13.1=1;\ \therefore\ 7.30-13.15=15.$$

Hence the general solution is $x=30+13m$, $y=15+7m$.

(2) We have $9.5-11.4=1$; $\therefore\ 9.20-11.16=4$.

Hence the general solution is $x=20+11m$, $y=16+9m$.

(3) We have $17 \cdot 7 - 15 \cdot 8 = -1$; $\therefore\ 17(-7 \cdot 31) - 15(-8 \cdot 31) = 31$.

Hence the general solution is $x = 15m - 217$, $y = 17m - 248$.

(4) We have $49 \cdot 31 - 69 \cdot 22 = 1$; $\therefore\ 49(3100) - 69(2200) = 100$.

Hence the general solution is

$$x = 3100 + 69m,\ y = 2200 + 49m.$$

4. (1) To solve $2x + 3y + 7z = 23$. Here z must be 1, 2, or 3. If $z = 3$, we should have $2x + 3y = 2$ which cannot be satisfied by positive integral values of x and y, zero values being excluded. If $z = 2$, we should have $2x + 3y = 9$ the only solution of which is $x = 3$, $y = 1$. If $z = 1$, we should have $2x + 3y = 16$ the general solution of which is given by $x = 3m - 16$, $y = 16 - 2m$, and the permissible values are $x = 2$, $y = 4$; $x = 5$, $y = 2$. Hence the solutions required are 3, 1, 2; 2, 4, 1; 5, 2, 1.

(2) To solve $7x + 4y + 18z = 109$. Here $7x + 4y \nmid 11$; $\therefore\ 18z \nmid 109 - 11$; the only values of z are therefore 1, 2, 3, 4 or 5. We then have

$$7x + 4y = 19,\ 7x + 4y = 37,\ 7x + 4y = 55,\ 7x + 4y = 73 \text{ and } 7x + 4y = 91.$$

The general solutions of these equations are $x = -19 + 4m$, $y = 38 - 7m$; $x = -37 + 4m$, $y = 74 - 7m$; $x = -55 + 4m$, $y = 110 - 7m$; $x = -73 + 4m$, $y = 146 - 7m$ and $x = -91 + 4m$, $y = 182 - 7m$.

(3) To solve $5x + y + 7z = 39$, $2x + 4y + 9z = 63$. Eliminating y we have $18x + 19z = 93$, the general solution of which is $x = -93 + 19m$, $z = 93 - 18m$; and then $y = 39 - 5x - 7z = -147 + 31m$. The only permissible value of m is 5, and we then have $x = 2$, $y = 8$, $z = 3$.

(4) To solve $3x + 2y + 3z = 250$, $9x - 4y + 5z = 170$. Eliminating y we have $15x + 11z = 670$, the general solution of which is $x = 2010 - 11m$, $z = -2680 + 15m$, and then $y = 1130 - 6m$. In order that x, y, z may be positive $m \nmid 182$ and $m \nmid 179$.

5. (1) We have $(x + 1)(2y - 3) = 1326$. Hence $x + 1$ is a factor of 1326, and the quotient must be odd. Hence $x + 1$ must be 2, 6, 26, 34, 78, 102, 442 or 1326; thus $x = 1$, $y = 333$; $x = 5$, $y = 112$; &c.

(2) We have $(x + 3)(x - y - 1) = 8$. Hence $x + 3$ must divide 8; and, if x be positive, we have $x + 3 = 8$, $x - y - 1 = 1$; or $x + 3 = 4$, $x - y - 1 = 2$.

(3) We have $(2x - 3y)(x + 4y) = 28$. Hence $x + 4y$ is a factor of 28; hence, as $x + 4y \nmid 5$, we have $x + 4y = 7$, $2x - 3y = 4$; or $x + 4y = 14$, $2x - 3y = 2$; or $x + 4y = 28$, $2x - 3y = 1$. Of these three pairs of equations the latter alone gives integral values, namely $x = 8$, $y = 5$.

(4) We have $(2x + y - 4)(x - y + 3) = 72$.

Hence the values of $2x + y - 4$ are 72, 36, 24, 18, 12, 9, 8, 6, 4, 3, 2, 1; the corresponding values of $x - y + 3$ being 1, 2, 3, 4, 6, &c. Hence $3x - 1$ must be equal to 73, 38, 27, 22, 18 or 17; and the only integral values of x are 13 and 6, the corresponding values of y being 14 and 1 respectively.

6. Since the integral values of y and z which satisfy $by+cz=k$ are in arithmetical progression; it follows that any one of x, y and z being supposed to be fixed, the possible values of either of the other two are in arithmetical progression.

If α, β, γ be any three values of x, y, z respectively which satisfy the equation, then the values $x=\alpha+(b-c)\,m$, $y=\beta+(c-a)\,m$, $z=\gamma+(a-b)\,m$ will satisfy the equation for all values of m.

7. Let $13x$ and $11y$ be the two parts; then $13x+11y=316$. The penultimate convergent to $\frac{11}{13}$ is $\frac{5}{6}$. Then $13(-5)+11\,.\,6=1$;

$$\therefore\ 13(-5\,.\,316)+11\,(6\,.\,316)=316.$$

Hence $$x=11m-1580 \text{ and } y=1896-13m.$$

Since x and y are to be positive, $m\not>145$ and $\not<144$. If $m=144$, $x=4$ and $y=24$; if $m=145$, $x=15$ and $y=11$. Hence the numbers are 52 and 264 or 195 and 121.

8. £1. 6*s*. 6*d*. = 53 sixpences. Hence we require the number of solutions in positive integers of $4x+5y=53$. The general solution is $x=-53+5m$, $y=53-4m$, the permissible values of m being 11, 12 and 13.

9. We require the number of solutions in positive integers of $21x+5y=2000$. The general solution is $x=2000-5m$, $y=21m-8000$, the permissible values of m ranging from 381 to 400 inclusive.

10. A must give x crowns to B and B must give y florins to A, so that $5x-2y=11$. The general solution is $x=11-2m$, $y=22-5m$. Hence, as $x\not>8$, the possible solutions are $x=7$, $y=12$; $x=5$, $y=7$ and $x=3$, $y=2$.

11. We have, if c be the number of sixpences in the money, $5x+4y=c$, the general solution of which can be put in the form $x=\alpha-4t$, $y=\beta+5t$, where $5\alpha+4\beta=c$, and α is the greatest possible value of x for that value of c. Now, in order that the number of solutions may be eight and eight only, the values of t must be 0, 1, 2, ..., 7; also β can have any of the values 1, 2, 3, 4, 5.

Hence, as *both* coins are to be used α must range from 29 to 32 inclusive and β from 1 to 5 inclusive. The least value of c is therefore

$$5\times29+4\times1=149,$$

the sum being £3. 14*s*. 6*d*.; also the greatest value of c is

$$5\times32+4\times5=180,$$

the sum being £4. 10*s*. 0*d*.

12. We have, if c be the number of pence in the sum, $4x=3y=c$, the general solution of which is $x=\alpha-3t$, $y=\beta+4t$, where $4\alpha+3\beta=c$, and α is the greatest possible value of x for that value of c.

Now, in order that the number of solutions may be three and three only, the values of t must be 0, 1 or 2; also β must be either 1, 2, 3 or 4.

Hence, as *both* coins are to be used α must be 7, 8 or 9. All the possible values of $4\alpha+3\beta$, when α is 7, 8 or 9 and β is 1, 2 or 3, are easily found to be 31, 34, 35, 37, 38, 39, 40, 41, 42, 44, 45 and 48.

13. $10x+y=kxy$. Hence $\frac{y}{x}$ is an integer. If $y=x$, $11=kx$ and therefore $x=1$, the number being 11. If $y=2x$, $12=2kx$, and therefore $x=1$, 2 or 3, the corresponding numbers being 12, 24, 36. If $y=3x$, $13=3kx$, which has no solution. If $y=4x$, $14=4kx$, which has no solution. If $y=5x$, $15=5kx$, and therefore $x=1$, the corresponding number being 15. If $y=6x$ or $y=7x$ or $y=8x$ there are no solutions.

14. We have $10x+\alpha=9\alpha+\beta$ and $10y+\alpha=9\beta+\alpha$. By subtraction $10(x-y)=8(\alpha-\beta)$; hence $\alpha-\beta=5$. If $\alpha=8$, $\beta=3$, the corresponding numbers being 75 and 35; if $\alpha=7$, $\beta=2$, the corresponding numbers being 65 and 25; and if $\alpha=6$, $\beta=1$, the corresponding numbers being 55 and 15.

15. Since the man's age is equal to the sum of four digits each of which is not greater than 9, the man must have been born in the 19th century, and the first two digits of the year of his birth are 1 and 8. Hence, if the other digits are x, y, we have

$$1+8+x+y=87-(10x+y);\ \therefore\ 11x+2y=78.$$

Since $y\not>9$, x must be 6 or 7. If $x=6$, $y=6$. If $x=7$, y is not integral. Thus the only values are $x=6$, $y=6$. The man's age is therefore

$$1+8+6+6=21.$$

16. In the product

$$(1+x^{a_1}+x^{2a_1}+\ldots)(1+x^{a_2}+x^{2a_2}+\ldots)(1+x^{a_3}+x^{2a_3}+\ldots)\ldots$$

where all the series go to infinity, the term involving x^m consists of all the products of terms $x^{x_1a_1}$, $x^{x_2a_2}$, &c., only one being taken out of each bracket, such that

$$x_1a_1+x_2a_2+\ldots=m \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

For every solution in positive integers of (i) there will correspond a term involving x^m in the above product, and therefore the number of solutions in positive integers of (i) is equal to the coefficient of x^m in the product.

The number of solutions of $x+2y=n$ is by the above equal to the coefficient of x^n in $\frac{1}{(1-x)(1-x^2)}$, that is in $\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1-x)^2}$, and this coefficient is equal to $\frac{1}{4}\{2n+3+(-1)^n\}$.

For the last part, we require the number of solutions of

$$x+2y+100z=20000.$$

This number is the coefficient of x^{20000} in $\dfrac{1}{(1-x)(1-x^2)(1-x^{100})}$, that is in

$$\frac{1}{(1-x)(1-x^2)}\{1+x^{100}+x^{200}+\ldots\}.$$

Hence, by the former case, the total number is

$$\frac{1}{4}(2\,.\,20000+3+1)+\frac{1}{4}(2\,.\,19900+3+1)+\ldots+\frac{1}{4}(2\,.\,130+3+1)+1=1005201.$$

17. We have $5x+3y+z=6000$. Hence $3y+z=5k$, where k has any value from 0 to 1200 inclusive.

The general solution of $3y+z=5k$ is $y=5k-m$, $z=3m-10k$; the number of solutions is therefore given from the conditions

$$m>5k \text{ and } m<\frac{10k}{3}.$$

Let $k=3l$; then l may have any value from 0 to 400 inclusive; and the number of solutions is

$$\Sigma_0^{400}(5l+1)=\frac{1}{2}\,.\,400\,(1+2001)=401401.$$

Let $k=3l+1$; then l may have any value from 0 to 399 inclusive; and the number of solutions is

$$\Sigma_0^{399}(5l+2)=\frac{1}{2}\,.\,400\,(2+1997)=399800.$$

Finally, let $k=3l+2$; then l may have any value from 0 to 399 inclusive; and the number of solutions is

$$\Sigma_0^{399}(5l+4)=\frac{1}{2}\,.\,400\,(4+1999)=400600.$$

Hence the total number of ways is $401401+399800+400600=1201802$.

PROBABILITY.

Art. 407. Ex. 2. There are three favourable and three unfavourable cases which are all equally likely. Hence the chance is $\dfrac{3}{6}=\dfrac{1}{2}$.

Ex. 5. The number of possible pairs of balls is ${}_5C_2=10$. The number of pairs of red balls is ${}_3C_2=3$. Thus there are 3 favourable and 7 unfavourable cases, which are all equally likely: hence the odds are 7 to 3 against the event.

Ex. 6. The total number of cases is ${}_6C_3 = 20$. The number of ways in which one ball of each colour can be drawn is $2 \cdot 2 \cdot 2 = 8$.

Hence the odds against drawing one ball of each colour are $20 - 8$ to 8 or 3 to 2.

Two white balls can be drawn in four ways, that is with each of the others. Hence the odds against drawing two white balls are $20 - 4$ to 4 or 4 to 1.

Ex. 7. Let A and B be the two specified persons. There are

$$\frac{1}{2}(n-1)(n-2)$$

pairs of persons which can sit next to A, and any one pair is as likely as any other to do so; also there are $n-2$ of these pairs which include B. Hence the chance that A and B sit together is $\frac{2}{n-1}$, the chance that they do not sit together is therefore $\frac{n-3}{n-1}$, and the odds against are $n-3$ to 2.

Art. 410. Ex. 2. With two dice 8 can be thrown in 5 ways, namely 2 and 6, 3 and 5, 4 and 4, 5 and 3, and 6 and 2. The total number of ways is 36. Hence the chance of throwing 8 is $\frac{5}{36}$.

Ex. 3. The number of ways of throwing 10 is the coefficient of x^{10} in

$$(x^1 + x^2 + \ldots + x^6)^2,$$

and this coefficient is 3. Hence the chance is $\frac{3}{36} = \frac{1}{12}$.

Ex. 4. The number of ways of throwing 15 with three dice is the coefficient of x^{15} in the expansion of $(x^1 + x^2 + \ldots + x^6)^3$, that is in

$$x^3 (1 - x^6)^3 (1 - x)^{-3},$$

and this coefficient is 10. The total number of throws is 6^3. Hence the chance is $\frac{10}{216} = \frac{5}{108}$.

Ex. 5. There are 36 possible throws with two dice, in 6 of which the numbers are the same. Hence the chance that A and B throw the same is $\frac{6}{36} = \frac{1}{6}$. The chance that they throw different numbers is $\frac{5}{6}$. Hence, as A and B are equally likely to throw the higher number, the chance that A's throw is higher than B is $\frac{5}{12}$.

Ex. 6. The number of ways of throwing 2, 3, &c. are the coefficients of the different powers of x in

$$(x^1 + x^2 + \ldots + x^6)^2 = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$$

Hence, as the total number of throws is 36, the chances that A throws 2, 3, &c. are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}$, &c. The chances that B throws 2, 3, &c. are the same. Hence the chance that their throws are equal is

$$\frac{1}{36}\cdot\frac{1}{36}+\frac{2}{36}\cdot\frac{2}{36}+\dots$$

$$=(1^2+2^2+3^2+4^2+5^2+6^2+5^2+4^2+3^2+2^2+1^2)/6^4=\frac{73}{648}.$$

Ex. 7. (1) In order that B may win, he must win the next two games, the chance of which is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. Hence the chance that B loses is $\frac{3}{4}$.

(2) In order that B may win, he must win the next three games, the chance of which is $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$. Hence the chance that B loses is $\frac{7}{8}$.

(3) The chance that A wins in two games is $\frac{1}{2}\cdot\frac{1}{2}$. The chance that A wins one and only one of the next two games is $\frac{1}{2}$, and then the game would be 5 to 3 and A's chance of winning would be $\frac{7}{8}$ by (2). The chance that A loses both the next two games played is $\frac{1}{2}\cdot\frac{1}{2}$, and then the game would be 4 all and A's chance of winning would be $\frac{1}{2}$. Hence A's chance of winning is

$$\frac{1}{4}+\frac{1}{2}\cdot\frac{7}{8}+\frac{1}{4}\cdot\frac{1}{2}=\frac{13}{16}.$$

Ex. 8. The match must be won in 4 games, and A will be the winner if he win two of these four games.

If a, b be the chances that A, B respectively win any particular game, the chance that A wins two or more out of 4 games is

$$a^4+4a^3b+6a^2b^2.$$

Hence, if $a=b=\frac{1}{2}$, the chance that A wins the match is $\frac{11}{16}$.

Ex. 9. The match must be finished in $2n$ games at most, and A will be the winner if he win n of these games. If a and b be the chances that A, B respectively win any particular game; the chance that A wins n games, at least, out of $2n$ is

$$a^{2n}+2na^{2n-1}b+\frac{\lfloor 2n}{\lfloor 2\ \lfloor n-2}a^{2n-2}b^2+\dots+\frac{\lfloor 2n}{\underline{n}\ \lfloor n}a^nb^n.$$

Put x for A's chance of winning the match and y for B's chance; then

$$x = a^{2n} + 2na^{2n-1}b + \ldots + \frac{\lfloor 2n}{\lfloor n \rfloor n} a^n b^n,$$

$$y = \frac{\lfloor 2n}{\lfloor n+1 \rfloor n-1} a^{n-1} b^{n+1} + \ldots + b^{2n}.$$

Hence, when $a = b = \frac{1}{2}$, we have, since $x + y = 1$,

$$x - y = \frac{\lfloor 2n}{\lfloor n \rfloor n} \frac{1}{2^{2n}} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n};$$

$$\therefore \; x = \frac{1}{2}\left\{1 + \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}\right\}$$

and

$$y = \frac{1}{2}\left\{1 - \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}\right\}.$$

Ex. 10. The chance is $\left(\frac{3}{5}\right)^3 + 3\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) = \frac{81}{125}$.

Ex. 11. The chance is

$$\left(\frac{2}{3}\right)^5 + \frac{5}{1}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + \frac{5 \cdot 4}{1 \cdot 2}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = \frac{192}{243}.$$

Ex. 12. The chance of throwing 6 on any particular occasion is $\frac{1}{6}$; hence the chance of throwing at least two sixes in 6 throws is

$$\left(\frac{1}{6}\right)^6 + 6\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right) + \frac{6 \cdot 5}{1 \cdot 2}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3$$
$$+ \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 = \frac{12281}{46656}.$$

Ex. 13. The following cases may occur:—

(1) the second and third throws may be the same as the first, and the chance of this is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$;

(2) the second throw may be the same and the third, fourth and fifth different from the first, and the chance of this is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$;

(3) the second, third and fourth may all be different from the first, and the chance of this is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$;

(4) the second throw may be different and the third, fourth and fifth throws may be the same as the first, and the chance of this is

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{16}.$$

Hence the chance that there may be three consecutive throws alike is

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{8}+\frac{1}{16}=\frac{1}{2}.$$

Ex. 14. The chance of any one winning the prize at a particular throw is $\frac{1}{2}\times$ the chance that he gets that throw. Hence the respective chances are

$$\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2^3}+\frac{1}{2}\cdot\frac{1}{2^6}+\ldots\text{to infinity}=\frac{4}{7},$$

$$\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2^4}+\frac{1}{2}\cdot\frac{1}{2^7}+\ldots\text{to infinity}=\frac{2}{7},$$

and

$$\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{2}\cdot\frac{1}{2^5}+\frac{1}{2}\cdot\frac{1}{2^8}+\ldots\text{to infinity}=\frac{1}{7}.$$

Art. 411. Ex. 2. (1) The chances of drawing 20*s.*, $2\frac{1}{2}s.$ or 1*s.* are respectively $\frac{2}{12}$, $\frac{3}{12}$ and $\frac{7}{12}$. Hence the expectation is

$$\left(20\cdot\frac{1}{6}+\frac{5}{2}\cdot\frac{1}{4}+\frac{7}{12}\right)s.=4s.\ 6\tfrac{1}{2}d.$$

(2) The sums in shillings that may be drawn are 40, $22\frac{1}{2}$, 21, 5, $3\frac{1}{2}$ and 2; and their respective chances are $\frac{1}{66}$, $\frac{6}{66}$, $\frac{14}{66}$, $\frac{3}{66}$, $\frac{21}{66}$ and $\frac{21}{66}$. Hence the expectation is

$$\frac{1}{66}\left(40+6\times22\tfrac{1}{2}+14\times21+3\times5+21\times3\tfrac{1}{2}+21\times2\right)s.=9s.\ 1d.$$

Ex. 3. The chance of the first is double that of the second. Hence the chances of winning the shilling are $\frac{2}{3}$ and $\frac{1}{3}$; and their expectations are therefore 8*d.* and 4*d.*

Ex. 4. The chance of the person who throws first is

$$\frac{1}{6}+\frac{1}{6}\left(\frac{5}{6}\right)^2+\frac{1}{6}\left(\frac{5}{6}\right)^4+\ldots\text{ to infinity;}$$

and the chance of the other is

$$\frac{1}{6}\left(\frac{5}{6}\right)+\frac{1}{6}\left(\frac{5}{6}\right)^3+\frac{1}{6}\left(\frac{5}{6}\right)^5+\ldots\text{ to infinity.}$$

Hence their chances are as 6 : 5, and therefore their expectations are 6*s.* and 5*s.* respectively.

XLI. Pages 526–529.

1. With two dice 8 may be thrown in 5 ways, and the whole number of throws is 36, so that the chance of throwing 8 on any particular occasion is $\frac{5}{36}$ and the chance of failing to throw 8 is $\frac{31}{36}$. Hence the chance of A's winning the prize the first time is $\frac{5}{36}$, of A's winning after both have failed once is $\frac{5}{6}\left(\frac{31}{36}\right)^2$, and so on. Thus A's chance is

$$\frac{5}{6}+\frac{5}{6}\left(\frac{31}{36}\right)^2+\frac{5}{6}\left(\frac{31}{36}\right)^4+\ldots=\frac{36}{67};$$

and B's chance is

$$\frac{5}{6}\left(\frac{31}{36}\right)+\frac{5}{6}\left(\frac{31}{36}\right)^3+\frac{5}{6}\left(\frac{31}{36}\right)^5+\ldots=\frac{31}{67}.$$

2. The number of ways in which 6 can be thrown with 3 dice is the coefficient of x^6 in $(x^1+x^2+x^3+\ldots+x^6)^3$, and this coefficient is 10.

Hence the chance of throwing 6 at any particular throw is $\frac{10}{6^3}$.

If A's chance is x, B's chance will be $x\times$ his chance of getting a throw, and C's chance is $x\times$ his chance of getting a throw.

Hence A's chance : B's chance : C's chance :: $1 : \left(1-\frac{10}{6^3}\right) : \left(1-\frac{10}{6^3}\right)^2$.

3. The chances of winning at the first, second, ..., sixth trial are respectively

$$\frac{3}{8},\ \frac{5}{8}\cdot\frac{3}{7},\ \frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6},\ \frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5},$$

$$\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4} \text{ and } \frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}.$$

Hence the chances of the three men are as

$$\frac{3}{8}+\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5} : \frac{5}{8}\cdot\frac{3}{7}+\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4} : \frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}+\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}.$$

4. The chances of throwing a six, 0, 1, 2, 3, &c. times in 50 throws with one die are the successive terms in the expansion of $\left(\frac{5}{6}+\frac{1}{6}\right)^{50}$. The terms increase up to the pth inclusive where p is the smallest integer for which $\frac{50-p+1}{p}\frac{1}{5}<1$. Hence the 9th term is the greatest, so that 8 sixes is the most likely number.

5. The chances of throwing 2, 3,..., 12 are the coefficients of x^2, x^3, &c. in $(x^1+x^2+\ldots+x^6)^2$.

Now x^7 is the middle term, and the coefficients are the same from the end as the beginning; hence the chance of throwing more than 7 is equal to the chance of throwing less than 7.

6. The chance of drawing an odd number is $\frac{2}{3}$ each time, and the chance of drawing an even number is $\frac{1}{3}$. The chances of drawing 4, 3, 2, 1, 0 odd numbers in four drawings are the terms of $\left(\frac{2}{3}+\frac{1}{3}\right)^4$. Hence the chance that the sum is odd is

$$4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)+4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3=\frac{40}{81},$$

and the chance that the sum is even is $\frac{41}{81}$.

7. Two odd numbers may be drawn in $\frac{1}{2}\cdot 50\cdot 49$ ways and two even numbers may be drawn in $\frac{1}{2}\cdot 50\cdot 49$ ways; also an odd and an even number may be drawn in 50^2 ways. Hence the sum may be even in $50\cdot 49$ ways and odd in $50\cdot 50$ ways, so that it is 50 : 49 that the sum is even.

8. The chance of drawing two tickets numbered p and q is $\frac{2}{n(n-1)}$, and the expectation is therefore $\frac{2}{n(n-1)}\Sigma pq$ shillings, where p and q are any two different numbers from 1 to n inclusive.

Now
$$2\Sigma pq=(1+2+3+\ldots n)^2-(1^2+2^2+\ldots+n^2)$$
$$=\frac{1}{12}n(n-1)(3n+2)(n+1).$$

Hence the expectation $=\frac{1}{12}(n+1)(3n+2)$ shillings.

9. The chance that a single event which is known to have happened in one or other of n years did not happen in a particular year is $\frac{n-1}{n}$.

Hence the chance that n such independent events did not happen in that particular year is $\left(\frac{n-1}{n}\right)^n$. But the same event happening n times is equivalent to n independent events.

10. As each thing can be given to any one of the p persons, there are p^p ways of distributing the p things without any restriction. But, if each person is to have one of the things, there are only $p!$ ways in which they can be distributed. Hence the number of ways in which one person at least is void is $p^p - p!$: the required chance is therefore $(p^p - p!)/p^p$.

11. Out of m^2N, a very large number, of letters sent by A to B, mN are lost and $m(m-1)N$ get to B. Of the $m(m-1)N$ answers sent by B to A, $(m-1)N$ are lost. Hence out of $(2m-1)N$ letters which are lost, mN are lost from A to B, and $(m-1)N$ from B to A. Hence the chance that B received the letter is $(m-1)/(2m-1)$.

12. Either 3 sovereigns and 1 shilling or 2 sovereigns and 2 shillings must have been put in the purse; and the à priori probabilities of these events will be found to be $\frac{1}{5}$ and $\frac{3}{5}$ respectively. Hence, with the notation of Art. 406, $P_1=\frac{1}{5}$. $P_2=\frac{3}{5}$; and $p_1=\frac{1}{2}$, $p_2=\frac{1}{6}$. Hence the probability that the purse really contained 3 sovereigns and 1 shilling is

$$P_1 \cdot p_1/(P_1 \cdot p_1 + P_2 \cdot p_2) = \frac{1}{2}.$$

Hence it is just as likely that the purse contained 3 sovereigns and 1 shilling as that it contained 2 sovereigns and 2 shillings. The value of the expectation of the remaining coins in the purse is therefore

$$\frac{1}{2}(21+2)s. = 11s.\ 6d.$$

13. There must have been either £4, £3. 1*s*. or £2. 2*s*. in the purse originally; and for these cases we have, as in Art. 406, $P_1=\frac{1}{70}$, $P_2=\frac{16}{70}$, $P_3=\frac{36}{70}$; also $p_1=1$, $p_2=\frac{1}{2}$, $p_3=\frac{1}{6}$. Hence the chances that the money left in the bag is 4*s*., 23*s*. or 42*s*. are in the ratio $\frac{1}{70}.1 : \frac{16}{70}\cdot\frac{1}{2} : \frac{36}{70}\cdot\frac{1}{6}$ that is $1:8:6$. Hence the value of the expectation is

$$\frac{6\times 42 + 8\times 23 + 4}{6+8+1} = 29s.\ 4d.$$

14. Let the points be A, B and C, and let A be taken at random on the circle. The chance that B and C both lie on the right of the diameter through A is $\frac{1}{4}$, and that they both lie on the left is $\frac{1}{4}$. Hence the chance that they are on the same side is $\frac{1}{2}$ and that they are on opposite sides is $\frac{1}{2}$. Let B and C be on opposite sides. Divide the circumference into $2n$ parts, and suppose that B can only fall on one of these points of division. We shall get the required chance by finding the chance on this hypothesis and making n infinite in the result.

The chance that B is r divisions from A is $\frac{1}{n}$, and in that case the chance that they are all on a semicircle is $\frac{n-r}{n}$. Hence, B and C being on opposite sides, the chance that they are all on a semicircle is $Lt_{n=\infty} \Sigma_{r=1}^{r=n} \frac{n-r}{n^2} = \frac{1}{2}$. Hence the required chance is $\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

15. Suppose the rod divided into $2n$ equal parts, where n is very great. Then the chance that one point of breakage is in the rth part from the middle is $\frac{2}{2n}$; and one of the three pieces will be greater than half the rod unless the other point of breakage is within $n-r$ of the middle and on the other side. Hence the chance that no one piece shall be greater than half the rod is the limit when n is infinite of $\Sigma_{r=1}^{r=n} \frac{2}{2n} \cdot \frac{n-r}{2n} = \frac{1}{4}$.

16. The rod has to be broken at three points taken at random. If one piece is greater than the sum of the other two it must be greater than half the rod. This will happen if all three points are on the same side of the centre. The chance of this is $\frac{1}{4}$. The chance that two points are on one side and one on the other is therefore $\frac{3}{4}$.

Imagine the rod divided into $2n$ parts and let two points of breakage on opposite sides of the centre be r and s parts respectively from the nearest ends. The chance that one part is greater than half the rod is then $\frac{2n-r-s}{2n}$, and the chance that r, s have any particular value is $\frac{1}{n^2}$. Hence, the points not being all on the same side, the chance that one part is greater than half the rod is

$$Lt_{n=\infty} \Sigma_{r=1}^{r=n} \Sigma_{s=1}^{s=n-r} \frac{2n-r-s}{2n^3} = \frac{1}{3}.$$

Hence the whole chance that one part is greater than half the rod is

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{2}.$$

17. If any one side has been chosen, there are $2(n-1)$ positions of the second side (excluding the case of two sides being parallel); and if the second side be the $(n+r)$th from the first, the third can only be one of those from the $(n+r+n+1)$th to the $(3n-1)$th counting from the first the same way round, so that when the second is the $(n+r)$th from the first, going round either way, the chance of the third side making an acute-angled triangle with the other two is $\frac{n-r-1}{4n-2}$. Hence the chance required is

$$\frac{1}{4n-1} \cdot 2\Sigma_{r=1}^{r=n-1} \frac{n-r-1}{4n-1} = \frac{(n-2)(n-1)}{(4n-1)(4n-2)}.$$

18. The chance that the second of the chosen numbers is next but one to the first is $\frac{2}{n-1}$, and the chance that it is neither next nor next but one to the first is $\frac{n-5}{n-1}$; and the chances that the third is not next to the first or to the second in these two cases are $\frac{n-5}{n-2}$ and $\frac{n-6}{n-2}$ respectively. Hence the chance required $= \frac{2}{n-1} \cdot \frac{n-5}{n-2} + \frac{n-5}{n-1} \cdot \frac{n-6}{n-2} = \frac{(n-4)(n-5)}{(n-1)(n-2)}$.

19. The n even integers can be written down in $\lfloor n$ ways; then, counting the outside places, there are $n+1$ places in which the m odd numbers are to be put only one being put in each place. The number of ways in which this can be done is

$$ {}_{n+1}P_m = \lfloor n+1 \,/\, \lfloor n-m+1. $$

The whole number of possible arrangements is $\lfloor m+n$.

Hence the required chance is $\frac{\lfloor n \, \lfloor n+1}{\lfloor n-m+1 \, \lfloor m+n}$.

20. The chance that any one thing goes to the men is $\frac{a}{a+b}$. Hence the chances that the men receive exactly 0, 1, 2, &c. things are the terms in the expansion of $\left(\frac{b}{a+b} + \frac{a}{a+b}\right)^m = C_0 + C_1 + C_2 + \ldots$

Now $\left(\frac{b}{a+b} - \frac{a}{a+b}\right)^m = C_0 - C_1 + C_2 + \ldots$

Hence $\frac{(b+a)^m - (b-a)^m}{(a+b)^m} = 2(C_1 + C_3 + C_5 + \ldots)$

$= 2 \times$ chance that the men receive an odd number of things.

21. One of the numbers is equally likely to be 1, 2, 3, ..., 99. The product is greater than 1000 provided one of the numbers is >11 and <89. Hence the chance is $\frac{77}{99}$.

22. Let the sum $= 4a$; then their greatest product $= 4a^2$. Let x be one part; then $x(4a-x) \neq 3a^2$, if $x^2 - 4ax + 3a^2 \neq 0$, that is if $(x-a)(x-3a) \neq 0$, which is the case when x lies between a and $3a$, the chance of which is one half.

Again, $x(4a-x) < 2a^2$, if $x^2 - 4ax + 2a^2 > 0$, that is if

$$ \{x - a(2+\sqrt{2})\}\{x - a(2-\sqrt{2})\} > 0, $$

which is the case if x is less than $(2-\sqrt{2})a$ or greater than $(2+\sqrt{2})a$. The chance is therefore $\frac{4a - 2\sqrt{2}a}{4a} = 1 - \frac{1}{\sqrt{2}}$.

23. Let u_n be A's chance of winning when he has n counters. The chance that he wins the next game is $\frac{1}{2}$, and his chance of ultimate success would then be u_{n+1}; also his chance of losing the next game is $\frac{1}{2}$, and his chance of ultimate success would then be u_{n-1}. Hence $u_n = \frac{1}{2}u_{n+1} + \frac{1}{2}u_{n-1}$, or $u_{n+1} - 2u_n + u_{n-1} = 0$. Hence u_n is the coefficient of x^n in $\frac{u_1 x}{(1-x)^2}$, since $u_0 = 0$; therefore $u_n = nu_1$. Hence $\frac{u_a}{a} = \frac{u_b}{b} = \frac{1}{a+b}$.

24. The question will not be altered if we suppose all the balls ranged in a line AB, the white ones to the left, the black to the right, the point of separation, X, being unknown and all positions for it in AB being *a priori* equally probable.

Then $p+q$ balls having been chosen at random, and p of these being white and q black, we know that AB is divided into $p+q+2$ sections, of which $p+1$ are white and $q+1$ are black, and that X is the $(p+1)$th point of division.

Now if one more trial be made, that is if one more point be taken at random, it is equally likely to fall on any one of the $p+q+2$ sections, and $p+1$ sections give white. Hence the chance that the next drawing is white is $\frac{p+1}{p+q+2}$. [Crofton.]

25. Call the side that wins the first point A, and the other B. When A has won x points and B has won y, the chance that the next two are won by A, B respectively is

$$\frac{(2m+1-x)(2m+1-y)}{(4m+2-x-y)(4m+1-x-y)},$$

and this is the chance that these games are won by B, A respectively. It follows from this that A's chance of winning p points out of q in any particular order is the same as for any other order.

Now, if A *just win* the game, he must score exactly $m-1$ points out of $2m$, and also win the last game. The chance that A wins $m-1$ games out of $2m-1$ in any particular order is the same as the chance that he loses the first m and wins the last $m-1$ games, and this latter chance is

$$\frac{2m+1}{4m+1} \cdot \frac{2m}{4m} \cdot \frac{2m-1}{4m-1} \cdots\cdots \frac{(m+2)}{3m+2} \cdot \frac{2m}{3m+1} \cdot \frac{2m-1}{3m} \cdots \frac{m+2}{2m+3}.$$

Hence, as there are ${}_{2m-1}C_{m-1}$ ways in which $m-1$ out of $2m-1$ may be won, and as the chance of A winning the last game when they have each won m games is $\frac{m+1}{2m+2}$, the chance required is

$$\frac{\lfloor 2m-1}{\lfloor m-1 \rfloor m} \cdot \frac{\{(2m+1)(2m)\ldots(m+2)\}\{2m(2m-1)\ldots(m+2)\}}{(4m+1)4m\ldots(3m+2)(3m+1)\ldots(2m+3)} \times \frac{m+1}{2m+2}$$

$$= \frac{\lfloor 2m-1 \rfloor 2m+1 \rfloor 2m+1 \rfloor 2m}{\lfloor m-1 \rfloor m \rfloor m+1 \rfloor m \rfloor 4m+1}.$$

[If 'will *just win*' means that A gets exactly m games out of the last $2m$ games, but not necessarily the last game of all, the chance will be as above but with ${}_{2m}C_m$ instead of ${}_{2m-1}C_{m-1}$, and this will give the result in the question.]

XLII. Pages 549—551.

1.
$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

$$= (b^2+c^2)\begin{vmatrix} c^2+a^2 & bc \\ cb & a^2+b^2 \end{vmatrix} - ab\begin{vmatrix} ba & bc \\ ca & a^2+b^2 \end{vmatrix} + ac\begin{vmatrix} ba & c^2+a^2 \\ ca & cb \end{vmatrix}$$

$$= a^2(b^2+c^2)(a^2+b^2+c^2) - a^2b^2(a^2+b^2-c^2) + a^2c^2(b^2-c^2-a^2) = 4a^2b^2c^2.$$

2. Subtract the first row from each of the others; then we have

$$\begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & (b-a)(a+b+c) \\ 0 & c-a & (c-a)(a+b+c) \end{vmatrix} = \begin{vmatrix} b-a & (b-a)(a+b+c) \\ c-a & (c-a)(a+b+c) \end{vmatrix} = 0.$$

3.
$$\begin{vmatrix} b+c & c+a & a+b \\ b'+c' & c'+a' & a'+b' \\ b''+c'' & c''+a'' & a''+b'' \end{vmatrix} = \begin{vmatrix} b & c & a \\ b' & c' & a' \\ b'' & c'' & a'' \end{vmatrix} + \begin{vmatrix} c & a & b \\ c' & a' & b' \\ c'' & a'' & b'' \end{vmatrix}$$

+ six other determinants each of which vanishes since two of its columns are identical.

4. Adding the second and third columns to the first, we have

$$\begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix} = 2(a+b+c)^3.$$

5. $$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ba & b^2+bc & c^2 \end{vmatrix} = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= abc \begin{vmatrix} a & c-a & 0 \\ a+b & -a & -2b \\ b & c & -2b \end{vmatrix} = abc \begin{vmatrix} a & c-a & 0 \\ a & -a-c & 0 \\ b & c & -2b \end{vmatrix} = 4a^2b^2c^2.$$

6. The determinant is equal to

$$(b+c)^2(bc+ca+ab)(bc+ca+ab+2a^2)+c^2(bc+ca+ab)(ab-bc-ca) + b^2(bc+ca+ab)(ca-ab-bc)$$

$$= (bc+ca+ab)^2\{(b+c)^2-b^2-c^2\}+\{2a^2(b+c)^2+2abc^2+2acb^2\}(bc+ca+ab)$$

$$= 2bc(bc+ca+ab)^2+2a(b+c)(bc+ca+ab)^2 = 2(bc+ca+ab)^3.$$

7. The determinant is equal to

$$-bc\{a^2bc-(ca+c^2)(ab+b^2)\}+b(b+c)\{ac(c+a)(a+b)+a^2b(a+c)\} + c(b+c)\{ab(c+a)(a+b)+a^2c(a+b)\}$$

$$= b^2c^2(bc+ca+ab)+ab(b+c)(c+a)(bc+ca+ab) + ac(b+c)(a+b)(bc+ca+ab)$$

$$= (bc+ca+ab)\{b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)\} = (bc+ca+ab)^3.$$

8. It is easy to see that the determinant would vanish if $a=0$. Hence a is a factor, and similarly b and c are factors.

Add the second and third rows to the first, and then the second and third columns to the first; then we have

$$\begin{vmatrix} 2(a+b+c)^2 & (b+c)(a+b+c) & (c+a)(a+b+c) \\ (b+c)(a+b+c) & (b+c)^2 & ab \\ (c+a)(a+b+c) & ab & (c+a)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} 2 & b+c & c+a \\ b+c & (b+c)^2 & ab \\ c+a & ab & (c+a)^2 \end{vmatrix}.$$

Hence $abc(a+b+c)^2$ is a factor of the determinant.

But the determinant is of the *sixth* degree and is symmetrical in a, b, c; the remaining factor must therefore be $(a+b+c)$. Hence the determinant is equal to $Labc(a+b+c)^3$, where L is found by putting $a=b=c=1$.

9. The determinant is equal to

$$(b+c)^2a(a+b+c)\{a(a+b+c)+2bc\}-a^2b^2(a+b+c)(a+b-c) - a^2c^2(a+b+c)(a-b+c)$$

$$= (a+b+c)^2\{a^2(b+c)^2-a^2b^2-a^2c^2\}+2abc(a+b+c)\{(b+c)^2+ab+ac\}$$

$$= 2a^2bc(a+b+c)^2+2abc(a+b+c)^2(b+c) = 2abc(a+b+c)^3.$$

10. Multiply the second, third and fourth columns of the first determinant by bc, ca, ab respectively; and then divide the rows by abc, a, b, c respectively.

This will not alter the value of the determinant. Hence

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}.$$

Consider the first determinant:—By adding all the rows to the first it is obvious that $a+b+c$ is a factor of the determinant. Adding the second row to the first and then subtracting the third and fourth rows, we see that $a-b-c$ is a factor of the determinant; and similarly $b-c-a$ and $c-a-b$ are seen to be factors. Hence as the determinant is of the fourth degree it is equal to $L(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$. The coefficient of a^4 in the determinant is clearly 1, and therefore $L=-1$.

11. Multiply the rows of the first determinant by $a\beta\gamma$, abc, $a\beta c$ and $ab\gamma$ respectively; and then divide the columns by abc, $a\beta\gamma$, $ab\gamma$ and $a\beta c$ respectively.

12. Subtract the third column from the fourth; then subtract the second column from the third; then subtract the first column from the second. We then have

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & -a & 0 \\ 1 & 0 & b & -b \\ 1 & 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & -a & 0 \\ 0 & b & -b \\ 0 & 0 & c \end{vmatrix} = abc.$$

13. As in 12, we have

$$\begin{vmatrix} 1+a & -a & 0 & 0 \\ 1 & b & -b & 0 \\ 1 & 0 & c & -c \\ 1 & 0 & 0 & d \end{vmatrix} = (1+a)\begin{vmatrix} b & -b & 0 \\ 0 & c & -c \\ 0 & 0 & d \end{vmatrix} + a\begin{vmatrix} 1 & -b & 0 \\ 1 & c & -c \\ 1 & 0 & d \end{vmatrix}$$

$$= (1+a)\,bcd + a(cd+bc+bd) = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

14. Add the second, third and fourth rows to the first: we then see that $a+b+c+d$ is a factor of the determinant.

Now add the second row and subtract the third and fourth rows from the first: we then see that $a+b-c-d$ is a factor of the determinant.

Add the third row and subtract the second and fourth rows from the first: we then see that $a+c-b-d$ is a factor of the determinant.

Lastly add the fourth row and subtract the second and third rows from the first: we then see that $a+d-b-c$ is a factor of the determinant.

Hence, as the determinant is of the fourth degree, it must be equal to

$$L(a+b+c+d)(a+b-c-d)(a+c-b-d)(a+d-b-c),$$

where L is numerical. The coefficient of a^4 in the determinant is clearly unity, and therefore $L=1$.

15. Add all the columns to the first; then we have

$$\begin{vmatrix} x+10 & x+10 & x+10 & x+10 \\ 1 & x+2 & 3 & 4 \\ 1 & 2 & x+3 & 4 \\ 1 & 2 & 3 & x+4 \end{vmatrix} = (x+10) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x+2 & 3 & 4 \\ 1 & 2 & x+3 & 4 \\ 1 & 2 & 3 & x+4 \end{vmatrix}$$

$$= (x+10) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x+1 & 2 & 3 \\ 0 & -x & x & 0 \\ 0 & 0 & -x & x \end{vmatrix} = (x+10) \begin{vmatrix} x+1 & 2 & 3 \\ -x & x & 0 \\ 0 & -x & x \end{vmatrix}$$

$$= (x+10)\{x^2(x+1)+2x^2-3x^2\} = x^3(x+10).$$

16. Multiply the columns in order by a, b, c, d, and then add the other three columns to the first; then we have

$$\frac{1}{abcd} \begin{vmatrix} a^2+b^2+c^2+d^2 & b^2 & c^2 & d^2 \\ 0 & ab & -cd & cd \\ 0 & db & ac & -bd \\ 0 & -cb & bc & ad \end{vmatrix}$$

$$= \frac{1}{a}(a^2+b^2+c^2+d^2) \begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

17. We have $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^3 & abc \end{vmatrix}.$

The second determinant will not be altered in value by multiplying its rows by a, b, c, d respectively, and then dividing the last column by $abcd$; we then have

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix} = 0.$$

18. $$\begin{vmatrix} a & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{vmatrix} = \begin{vmatrix} a & 0 & 0 & 0 \\ a & b-a & a-b & 0 \\ a & 0 & b-a & a-b \\ a & 0 & 0 & b-a \end{vmatrix}$$

$$= a \begin{vmatrix} b-a & a-b & 0 \\ 0 & b-a & a-b \\ 0 & 0 & b-a \end{vmatrix} = a(b-a)^3.$$

19. $$\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ b & b & a & b \\ a & a & a & b \end{vmatrix} = \begin{vmatrix} a & b-a & 0 & 0 \\ a & b-a & a-b & 0 \\ b & 0 & a-b & b-a \\ a & 0 & 0 & b-a \end{vmatrix}$$

$$= a \begin{vmatrix} b-a & a-b & 0 \\ 0 & a-b & b-a \\ 0 & 0 & b-a \end{vmatrix} - (b-a) \begin{vmatrix} a & a-b & 0 \\ b & a-b & b-a \\ a & 0 & b-a \end{vmatrix}$$

$$= a(a-b)^3 - (b-a)\{-a(a-b)^2 - (a-b)^3\} = -(a-b)^4.$$

20. Having multiplied the columns in order by a, b, c respectively, and the rows by x, y, z respectively, add the second and third columns to the first, and then add the second and third rows to the first; we thus have

$$\frac{1}{abcxyz} \begin{vmatrix} (a^2+b^2+c^2)(x^2+y^2+z^2) & b^2(x^2+y^2+z^2) & c^2(x^2+y^2+z^2) \\ (a^2+b^2+c^2)y^2 & by(by-cz-ax) & cy(bz+cy) \\ (a^2+b^2+c^2)z^2 & bz(bz+cy) & cz(cz-ax-by) \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{abcxyz} \begin{vmatrix} 1 & b^2 & c^2 \\ y^2 & by(by-cz-ax) & cy(bz+cy) \\ z^2 & bz(bz+cy) & cz(cz-ax-by) \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2) \begin{vmatrix} 1 & b & c \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2) \begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)\{(ax+by)(ax+cz) - bcyz\}$$

$$= (a^2+b^2+c^2)(x^2+y^2+z^2)(ax+by+cz).$$

21. Subtract the first row from each of the others; then we have

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2-a^2 & 2(b-a)(a+b-c) & c(a-b) \\ c^2-a^2 & 2(c-a)(a-b+c) & b(a-c) \end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b+a & 2a+2b-2c & -c \\ c+a & 2a-2b+2c & -b \end{vmatrix}.$$

Subtract four times the third column from the second, and then subtract the second row from the third; we then have

$$=(b-a)(c-a)\begin{vmatrix} a^2 & a^2-(b+c)^2 & bc \\ b+a & 2a+2b+2c & -c \\ c-b & 0 & c-b \end{vmatrix}$$

$$=(b-c)(c-a)(a-b)(a+b+c)\begin{vmatrix} a^2 & a-b-c & bc \\ b+a & 2 & -c \\ 1 & 0 & 1 \end{vmatrix}$$

$$=(b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2).$$

22. The determinant is equal to

$$3(b-c)^2(c-a)^2(a-b)^2-(b-c)^6-(c-a)^6-(a-b)^6$$

$$=-\{(b-c)^2+(c-a)^2+(a-b)^2\}\{(b-c)^4+(c-a)^4+(a-b)^4$$
$$-(b-c)^2(c-a)^2-(c-a)^2(a-b)^2-(a-b)^2(b-c)^2\}$$

$$=-\frac{1}{4}\{(b-c)^2+(c-a)^2+(a-b)^2\}^3.$$

23. Let the determinant be

$$\begin{vmatrix} a_1, & a_2, & a_3, \ldots\ldots, & a_n \\ b_1, & b_2, & b_3, \ldots\ldots, & b_n \\ c_1, & c_2, & c_3, \ldots\ldots, & c_n \\ & \ldots\ldots\ldots\ldots & & \end{vmatrix}.$$

Then, since the determinant vanishes, we have

$$a_1 . A_1+a_2 . A_2+a_3 . A_3+\ldots+a_n . A_n=0.$$

We have also [Art. 420]

$$c_1 . A_1+c_2 . A_2+c_3 . A_3+\ldots+c_n . A_n=0,$$
$$d_1 . A_1+d_2 . A_2+d_3 . A_3+\ldots+d_n . A_n=0.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

But these $n-1$ equations are all true when $B_1, B_2, \ldots$ are substituted for $A_1, A_2, \ldots$ respectively; and therefore $A_1 : B_1 = A_2 : B_2 = \ldots$

24. $\Delta = abcd \begin{vmatrix} a+\frac{1}{a} & b & c & d \\ a & b+\frac{1}{b} & c & d \\ a & b & c+\frac{1}{c} & d \\ a & b & c & d+\frac{1}{d} \end{vmatrix} = abcd \begin{vmatrix} a+\frac{1}{a} & b & c & d \\ -\frac{1}{a} & \frac{1}{b} & 0 & 0 \\ 0 & -\frac{1}{b} & \frac{1}{c} & 0 \\ 0 & 0 & -\frac{1}{c} & \frac{1}{d} \end{vmatrix}$

$$= a^2+1+b^2+c^2+d^2.$$

25. This follows at once from the multiplication of

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & \alpha \\ 0 & 1 & b & \beta \\ 0 & 1 & c & \gamma \end{vmatrix} \text{ and } \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & a & \alpha \\ 1 & 0 & b & \beta \\ 1 & 0 & c & \gamma \end{vmatrix}.$$

26. Add twice the first row to the last, and the second and third to the fourth; we then have

$$\begin{vmatrix} 0 & 0 & 0 & a & b & c \\ 0 & 0 & z & a & b & 0 \\ 0 & y & 0 & a & 0 & c \\ x & y & z & 2a & 2b & 2c \\ x & y & z & 2a & 2b & 2c \end{vmatrix}.$$

Since two rows are now alike it follows that all the determinants vanish.

27. Multiply the first column by x, and then subtract from it y times the second column and z times the third; we then have

$$\frac{1}{x} \begin{vmatrix} x^3+y^3+z^3-3xyz & y^2-zx & z^2-xy \\ 0 & x^2-yz & y^2-zx \\ 0 & z^2-xy & x^2-yz \end{vmatrix}$$

$$= \frac{1}{x}(x^3+y^3+z^3-3xyz) \begin{vmatrix} x^2-yz & y^2-zx \\ z^2-xy & x^2-yz \end{vmatrix} = (x^3+y^3+z^3-3xyz)^2.$$

Also $$\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = 3xyz - x^3 - y^3 - z^3.$$

28. Forming the product of the two determinants on the left, we have

$$\begin{vmatrix} \lambda^3+(a^2+b^2+c^2)\lambda, & 0 & , & 0 \\ 0 & , & \lambda^3+(a^2+b^2+c^2)\lambda, & 0 \\ 0 & , & 0 & , & \lambda^3+(a^2+b^2+c^2)\lambda \end{vmatrix}.$$

29. Subtract the first column from the last and the second from the third; then we have

$$\begin{vmatrix} x & y & z-y & w-x \\ a & b & c-b & d-a \\ d & c & b-c & a-d \\ w & z & y-z & x-w \end{vmatrix} = \frac{1}{4} \begin{vmatrix} 2x & 2y & z-y & w-x \\ 2a & 2b & c-b & d-a \\ 2d & 2c & b-c & a-d \\ 2w & 2z & y-z & x-w \end{vmatrix}.$$

Now add the fourth column to the first and the third to the second; then we have

$$\frac{1}{4} \begin{vmatrix} x+w & y+z & z-y & w-x \\ a+d & b+c & c-b & d-a \\ a+d & b+c & b-c & a-d \\ a+w & y+z & y-z & x-w \end{vmatrix} = \frac{1}{4} \begin{vmatrix} x+w & y+z & z-y & w-x \\ a+d & b+c & c-b & d-a \\ 0 & 0 & 2(b-c) & 2(a-d) \\ 0 & 0 & 2(y-z) & 2(x-w) \end{vmatrix}$$

$$= \begin{vmatrix} x+w & y+z \\ a+d & b+c \end{vmatrix} \cdot \begin{vmatrix} b-c & a-d \\ y-z & x-w \end{vmatrix}.$$

XLIII. Pages 564—565.

1. (i) Since $a_1+a_2+a_3=0,\ a_2+a_3=-a_1,$ &c.

Hence $(a_2+a_3)(a_3+a_1)(a_1+a_2)=-a_1a_2a_3=q.$

(ii) $a_2+a_3-2a_1=-3a_1,$ &c.

Hence we have $-27a_1a_2a_3=27q.$

(iii) $\Sigma a_1^2=(\Sigma a_1)^2-2\Sigma a_1a_2=-2p.$

(iv) $\Sigma a_1^3-3a_1a_2a_3=(a_1+a_2+a_3)(\Sigma a_1^2-\Sigma a_1a_2)=0;$

$\therefore\ \Sigma a_1^3=3a_1a_2a_3=-3q.$

(v) We have $a_1^3+pa_1+q=0,$ &c.

Hence $a_1^4+pa_1^2+qa_1=0,$ &c.

Hence $\Sigma a_1^4+p\Sigma a_1^2=0.$

$\therefore\ \Sigma a_1^4=-p\Sigma a_1^2=-p\{(\Sigma a_1)^2-2\Sigma a_1a_2\}=2p^2.$

(vi) $\Sigma a_1^2a_2=\Sigma a_1a_2\,.\,\Sigma a_1-3\Sigma a_1a_2a_3=-3\Sigma a_1a_2a_3=3q.$

(vii) $\Sigma a_1^3a_2=\Sigma a_1a_2\,.\,\Sigma a_1^2-\Sigma a_1^2a_2a_3$

$=\Sigma a_1a_2\{(\Sigma a_1)^2-2\Sigma a_1a_2\}-a_1a_2a_3\Sigma a_1=-2p^2.$

(viii) $a_1^2-a_2a_3=-a_1(a_2+a_3)-a_2a_3=-p.$

Hence the given expression $=3p^2.$

(ix) As in viii, $a_1^2-a_2a_3=-p,$ &c.

Hence the given expression $=(-p)^3.$

(x) $\Sigma\dfrac{1}{a_2+a_3}=-\Sigma\dfrac{1}{a_1}=-\dfrac{\Sigma a_1a_2}{a_1a_2a_3}=\dfrac{p}{q}.$

(xi) $\Sigma\dfrac{1}{a_2+a_3-a_1}=-\dfrac{1}{2}\Sigma\dfrac{1}{a_1}=\dfrac{p}{2q}.$

(xii) $\Sigma\dfrac{1}{a_1^2+a_2a_3}=\Sigma\dfrac{1}{-a_1(a_2+a_3)+a_2a_3}=\Sigma\dfrac{1}{2a_2a_3-p}$

$=\dfrac{\Sigma(2a_1a_2-p)(2a_1a_3-p)}{(2a_2a_3-p)(2a_3a_1-p)(2a_1a_2-p)}$

$=\dfrac{4a_1a_2a_3\Sigma a_1-4p\Sigma a_1a_2+3p^2}{8a_1^2a_2^2a_3^2-4pa_1a_2a_3\Sigma a_1+2p^2\Sigma a_1a_2-p^3}$

$=\dfrac{-4p^2+3p^2}{8q^2+2p^3-p^3}=-\dfrac{p^2}{8q^2+p^3}.$

2. Here $\Sigma a_1 = 0,\ \Sigma a_1 a_2 = 0,\ \Sigma a_1 a_2 a_3 = -p,\ a_1 a_2 a_3 a_4 = q.$

Hence $\Sigma a_1^2 = (\Sigma a_1)^2 - 2\Sigma a_1 a_2 = 0$ .....(i),

$\Sigma a_1^3 - 3\Sigma a_1 a_2 a_3 = \Sigma a_1 \{\Sigma a_1^2 - \Sigma a_1 a_2\}$;

$\therefore\ \Sigma a_1^3 = 3\Sigma a_1 a_2 a_3 = -3p$.....(ii).

Also $a_1^4 + p a_1 + q = 0$, &c.

Hence, by addition, $\Sigma a_1^4 + p\Sigma a_1 + 4q = 0$;

$\therefore\ \Sigma a_1^4 = -4q.$

3. (i) $b + c - 3a = -p - a - 3a$, &c.

Hence, we have $-(p+4a)(p+4b)(p+4c)$

$= -p^3 - 4p^2(a+b+c) - 16p\Sigma ab - 64abc$

$= -p^3 - 4p^2(-p) - 16pq + 64r.$

(ii) $\dfrac{1}{b} + \dfrac{1}{c} - \dfrac{1}{a} = \dfrac{1}{abc}(ab + ac - bc) = \dfrac{1}{abc}(q - 2bc)$, &c.

Hence we have $\dfrac{1}{a^3b^3c^3}(q-2bc)(q-2ca)(q-2ab)$

$= \dfrac{1}{a^3b^3c^3}\{q^3 - 2q^2\Sigma bc + 4qabc\Sigma a - 8a^2b^2c^2\}$

$= -\dfrac{1}{r^3}\{q^3 - 2q^3 + 4qrp - 8r^2\}.$

(iii) Since $bc - a^2 = bc - a(-p-b-c) = \Sigma bc + ap = q + ap$, &c.

The given expression $= (q+ap)(q+bp)(q+cp)/a^4b^4c^4$

$= (q^3 + q^2p\Sigma a + qp^2\Sigma bc + abcp^3)/a^4b^4c^4$

$= (q^3 - q^2p^2 + q^2p^2 - rp^3)/r^4.$

4. (i) $\Sigma a_1 = 0,\ \Sigma a_1 a_2 = -14,\ a_1 a_2 a_3 = -8.$

Hence $\Sigma a_1^2 = (\Sigma a_1)^2 - 2\Sigma a_1 a_2 = 28,$

$\Sigma a_1^3 = 3a_1 a_2 a_3 + \Sigma a_1(\Sigma a_1^2 - \Sigma a_1 a_2) = -24.$

(ii) $\Sigma a_1 = 0,\ \Sigma a_1 a_2 = -22,\ \Sigma a_1 a_2 a_3 = -84.$

Hence $\Sigma a_1^2 = (\Sigma a_1)^2 - 2\Sigma a_1 a_2 = 44.$

Also $\Sigma a_1^3 - 3\Sigma a_1 a_2 a_3 = \Sigma a_1(\Sigma a_1^2 - \Sigma a_1 a_2)$;

$\therefore\ \Sigma a_1^3 = 3\Sigma a_1 a_2 a_3 = -252.$

5. (i) $\Sigma a^2 = (\Sigma a)^2 - 2\Sigma ab = (-p_1)^2 - 2p_2.$

(ii) $\Sigma a^3 - 3\Sigma abc = \Sigma a\{\Sigma a^2 - \Sigma ab\} = \Sigma a\{(\Sigma a)^2 - 3\Sigma ab\}$;

$\therefore\ \Sigma a^3 = -3p_3 - p_1\{p_1^2 - 3p_2\}.$

(iii) $\Sigma \dfrac{1}{a^2} = \left(\Sigma \dfrac{1}{a}\right)^2 - 2\Sigma \dfrac{1}{ab} = \left(-\dfrac{p_{n-1}}{p_n}\right)^2 - 2\dfrac{p_{n-2}}{p_n}.$

(iv) Since $$\Sigma a^2 \cdot \Sigma \frac{1}{a} = \Sigma a + \Sigma \frac{a^2}{b},$$

$$\Sigma \frac{a^2}{b} = -(p_1^2 - 2p_2)\frac{p_{n-1}}{p_n} + p_1.$$

(v) $\Sigma \frac{a^3}{b} = \Sigma a^3 \cdot \Sigma \frac{1}{a} - \Sigma a^2$. Hence from (i) and (ii).

(vi) $\Sigma \frac{a^3}{b^2} = \Sigma a^3 \cdot \Sigma \frac{1}{a^2} - \Sigma a$. Hence from (ii) and (iii).

6. Put $x+2=y$, or $x=y-2$.

Then $$(y-2)^3 - 4(y-2)^2 + 3(y-2) - 1 = 0,$$
$$\text{i.e. } y^3 - 10y^2 + 31y - 31 = 0.$$

7. Put $cx=y$, or $x=\frac{y}{c}$.

Then $$6y^3/c^3 - 5y^2/c^2 - \frac{1}{4} = 0;$$

$$\therefore y^3 - \frac{5}{6}cy^2 - \frac{c^3}{24} = 0.$$

The coefficients are all integral if $c=6$.

8. (i) Put $y=bc=-\frac{r}{a}$; then $a=-\frac{r}{y}$.

Hence, as a is a root of the given equation, the required equation will be

$$-\frac{r^3}{y^3} + \frac{pr^2}{y^2} - \frac{qr}{y} + r = 0,$$

or $$y^3 - qy^2 + pry - r^2 = 0.$$

(ii) Put $y=b+c=-p-a$.

Hence the equation required is

$$-(p+y)^3 + p(p+y)^2 - q(p+y) + r = 0.$$

(iii) Put $y=\frac{a}{b+c}=\frac{a}{-p-a}$; $\therefore a=\frac{-py}{1+y}$.

Hence the required equation is

$$-\frac{p^3y^3}{(1+y)^3} + p\frac{p^2y^2}{(1+y)^2} - q\frac{py}{1+y} + r = 0.$$

(iv) Put $y=a(b+c)=a(-p-a)$.

Hence $$a^2 + ap + y = 0.$$

Also $$a^3 + a^2p + aq + r = 0.$$

Multiply the first equation by a and subtract from the second; then

$$a(q-y)+r=0.$$

Hence the equation required is

$$\frac{r^2}{(y-q)^2}+\frac{pr}{y-q}+y=0,$$

which reduces to

$$y^3-2qy^2+(q^2+pr)\,y+r^2-pqr=0.$$

(v) Put $$y=(b^2+c^2)=(b+c)^2-2bc=(p+a)^2+\frac{2r}{a}.$$

Hence $$a^3+2a^2p+(p^2-y)\,a+2r=0.$$

Also $$a^3+a^2p+aq+r=0.$$

Subtract; then $$a^2p+(p^2-q-y)\,a+r=0.$$

Multiply the second equation by 2, and subtract from the first; then

$$-a^2+p^2-2q-y=0.$$

Then eliminate a between the last two equations.

(vi) Put $$y=bc-a^2=-\frac{r}{a}-a^2.$$

Then $$a^3+ay+r=0.$$

Also $$a^3+a^2p+aq+r=0.$$

Hence $$a^2p+a(q-y)=0, \text{ whence } a=(y-q)/p.$$

Hence the equation required is

$$(y-q)^3/p^3+y\,(y-q)/p+r=0.$$

9. (i) Put $y=b+c+d=-p-a$; then $a=-p-y$.

Hence the equation required is

$$(p+y)^4-p\,(p+y)^3+q\,(p+y)^2-r\,(p+y)+s=0.$$

(ii) Put $$y=b+c+d-2a=-p-3a;$$

then $$a=-\frac{1}{3}(p+y). \quad \text{Hence \&c.}$$

(iii) $$\Sigma a^2=(\Sigma a)^2-2\Sigma ab=p^2-2q.$$

Hence, if we put

$$y=b^2+c^2+d^2-a^2=\Sigma a^2-2a^2=p^2-2q-2a^2$$

we shall have to eliminate a from

$$2a^2+y+2q-p^2=0,$$

and $$a^4+pa^3+qa^2+ra+s=0.$$

From the last equation we have

$$(a^4+qa^2+s)^2=a^2(pa^2+r)^2.$$

The result is obtained by substituting

$$a^2=\frac{1}{2}(p^2-2q-y).$$

10. Since $(x^3+r)+px^2+qx=0,$

$$(x^3+r)^3+(px^2)^3+(qx)^3-3(x^3+r)\,.\,px^2\,.\,qx=0.$$

Now put $x^3=y$, and we have

$$(y+r)^3+p^3y^2+q^3y-3qpy(y+r)=0,$$

which is the equation required.

XLIV. Pages 569—570.

1. Corresponding to the *two* roots $2\pm\sqrt{3}$ we have the factor

$$(x-2-\sqrt{3})(x-2+\sqrt{3}),\text{ that is } x^2-4x+1.$$

Hence the given equation is

$$(x^2-4x+1)(x^2+6x+7)=0.$$

Thus the roots are $2\pm\sqrt{3},\ -3\pm\sqrt{2}$.

2. Corresponding to the *two* roots $3\pm\sqrt{-5}$ we have the factor

$$(x-3-\sqrt{-5})(x-3+\sqrt{-5}),\text{ that is } x^2-6x+14.$$

Hence the given equation is

$$(x^2-6x+14)(3x-5)=0.$$

3. The equation must have the *four* roots $\pm\sqrt{2}\pm\sqrt{5}$; and, the sum of these roots being zero, the remaining roots must be $\frac{4}{3}$.

4. The equation has the four roots $\pm\sqrt{2}\pm\sqrt{-1}$, corresponding to which is the factor x^4-2x^2+9. Thus the given equation is

$$(x^4-2x^2+9)(2x^2-3x+9)=0.$$

5. The required equation is

$$(x-\sqrt{3}-\sqrt{5})(x-\sqrt{3}+\sqrt{5})(x+\sqrt{3}-\sqrt{5})(x+\sqrt{3}+\sqrt{5})=0.$$

6. The required equation is

$$(x-\sqrt{2}-\sqrt{-3})(x-\sqrt{2}+\sqrt{-3})(x+\sqrt{2}-\sqrt{-3})(x+\sqrt{2}+\sqrt{-3})=0.$$

7. The H.C.F. of the left-hand members will be found to be x^2-3x+1. Hence the equations have two roots in common, namely the roots of

$$x^2-3x+1=0.$$

8. If two of the roots are $\alpha+\beta\sqrt{-1}$ and $\alpha+2\beta\sqrt{-1}$, the other two will be $\alpha-\beta\sqrt{-1}$ and $\alpha-2\beta\sqrt{-1}$. Hence the equation is

$$\{(x-\alpha)^2+\beta^2\}\{(x-\alpha)^2+4\beta^2\}=0,$$

or

$$(x-\alpha)^4+5\beta^2(x-\alpha)^2+4\beta^4=0,$$

i.e. $x^4-4\alpha x^3+(6\alpha^2+5\beta^2)x^2+\ldots=0.$

Hence $4\alpha=4$ and $6\alpha^2+5\beta^2=11,$

whence it follows that $\alpha=1=\beta$.

9. If the roots α, β, γ be in H.P.,

$$\frac{3}{\beta}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=-\frac{q}{r}.$$

Hence $$\beta=-\frac{3r}{q}.$$

But β is a root of the original equation; hence the required condition is

$$\left(-\frac{3r}{q}\right)^3+p\left(-\frac{3r}{q}\right)^2+q\left(-\frac{3r}{q}\right)+r=0.$$

10. Any four quantities in A.P. may be taken to be

$$a+3d,\ a+d,\ a-d,\ a-3d.$$

Hence
$$-p=4a,$$
$$q=6a^2-10d^2,$$
$$r=4a^3-20ad^2,$$
$$s=a^4-10a^2d^2+9d^4.$$

From the first two equations we have

$$a=-\frac{1}{4}p,\quad d^2=\frac{1}{80}(3p^2-8q).$$

Substitute for a and d^2 in the last two equations.

11. If α, β, γ be in A.P.,

$$3\beta=\alpha+\beta+\gamma=3.$$

Thus the mean root is 1. Hence the equation is

$$(x-1)(x^2-2x-15)=0,$$

and the roots are $-3, 1, 5$.

12. Let the roots be $a+3d, a+d, a-d, a-3d$.

Then $$4a=-2, \text{ and } (a^2-d^2)(a^2-9d^2)=40,$$

whence $$a=-\frac{1}{2},\ d=\pm\frac{3}{2}.$$

13. (i) $$x^3-7x^2+17x-15=0.$$

The positive commensurable roots can only be 15, 5, 3, 1. Of these 3 is the only value which satisfies the equation.

There can be no real negative root, for the left-hand member of the equation is negative for every negative value of x.

(ii) $$x^4-x^3-13x^2+16x-48=0.$$

The possible commensurable roots are $\pm48, \pm24, \pm16, \pm8, \pm6, \pm4, \pm3, \pm2, \pm1$. Of these ±4 are the only values which satisfy the equation.

(iii) $$3x^3-26x^2+34x-12=0.$$

The possible commensurable roots are 4, 2, 1 or $\frac{4}{3}, \frac{2}{3}, \frac{1}{3}$ for there can be no negative roots.

The only value which satisfies the equation is $\frac{2}{3}$.

14. Since the sum of two roots is zero, the other root must be 8. Hence the equation is

$$(x-8)(4x^2-1)=0.$$

15. Let the four roots be α, β, γ, δ, and let $\alpha+\beta=0$.

Then
$$-4=\alpha+\beta+\gamma+\delta=\gamma+\delta$$
$$-5=\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta)=\alpha\beta+\gamma\delta$$
$$8=\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=\alpha\beta(\gamma+\delta)$$
$$6=\alpha\beta\gamma\delta.$$

Hence $\alpha\beta=-2$, $\gamma\delta=-3$.

Then $\alpha+\beta=0$, $\alpha\beta=-2$ give $\alpha=-\beta=\sqrt{2}$,

and $\gamma+\delta=-4$, $\gamma\delta=-3$ give $\gamma=-2+\sqrt{7}$, $\delta=-2-\sqrt{7}$.

16. We have, if
$$\alpha+\beta=0,$$
$$-p=\gamma+\delta,$$
$$q=\alpha\beta+\gamma\delta,$$
$$-r=\alpha\beta(\gamma+\delta),$$
$$s=\alpha\beta\gamma\delta.$$

Hence $\alpha\beta=\dfrac{r}{p}$ and then

$$\gamma\delta=q-\frac{r}{p}=\frac{sp}{r}.$$

Thus the condition required is

$$pqr-r^2-sp^2=0.$$

17. Since α is a root
$$(2\beta+1)^3-79(2\beta+1)+210=0,$$
$$\text{i.e. } 2\beta^3+3\beta^2-38\beta+33=0.$$

Also
$$\beta^3-79\beta+210=0.$$

The common root of the last two equations is $\beta=3$.

Hence two of the roots are 3, 7 and the remaining root must be -10.

18. If $\alpha=\beta^2$, we have
$$3\beta^6-32\beta^4+33\beta^2+108=0,$$

and
$$3\beta^3-32\beta^2+33\beta+108=0.$$

The common root of these equations could be found to be 3. [This laborious process may be avoided by noticing that β cannot be of the form $a+\sqrt{b}$, for then there would be *four* roots of the equation, namely $a\pm\sqrt{b}$, $a^2+b\pm2a\sqrt{b}$. Also, if $\beta=\sqrt{b}$ and $\alpha=b$, the remaining root would $-\sqrt{b}$; the sum of the roots would be b and the continued product $-b^2$. We should therefore have $b=\dfrac{32}{3}$ and $b^2=36$, which are inconsistent. *Hence β is altogether rational*, and if $\beta=\dfrac{p}{q}$, $\alpha=\dfrac{p^2}{q^2}$ be roots it follows that p^3 is a factor of 108 and q^3 of 3, whence $q=1$ and $p=\pm3$. The value $\beta=3$ satisfies the equation.]

19. Let the roots be a, $a+d$, $a+2d$, &c.

Then the sum of the roots is $na+\frac{1}{2}n(n-1)d$,

and therefore $na+\frac{1}{2}n(n-1)d=-np$, or $a+\frac{1}{2}(n-1)d=-p$ ..... (i).

Also $\{a+(a+d)+(a+2d)+\ldots\}^2=\Sigma(a+rd)^2+2\Sigma(a+rd)(a+sd)$.

$$\therefore \left\{na+\frac{1}{2}n(n-1)d\right\}^2=a^2+(a+d)^2+(a+2d)^2+\ldots+n(n-1)q$$

$$=na^2+n(n-1)ad+\frac{1}{6}(n-1)n(2n-1)d^2+n(n-1)q,$$

whence $a^2+(n-1)ad+\frac{1}{4}n(n-1)d^2-\frac{1}{6}(2n-1)d^2=q$ ..... (ii).

From (i) and (ii), we have

$d=2\{3(p^2-q)/(n+1)\}^{\frac{1}{2}}$ and $a=-p-(n-1)\{3(p^2-q)/(n+1)\}^{\frac{1}{2}}$.

20. We have

$$\alpha+\beta+\gamma+\delta=-p,$$
$$\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta)=q,$$
$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-r,$$
$$\alpha\beta\gamma\delta=s.$$

Hence, if $\alpha\beta=\gamma\delta$, we have

$$\alpha\beta=\gamma\delta=\sqrt{s},$$
$$\sqrt{s}(\alpha+\beta+\gamma+\delta)=-r;$$
$$\therefore p\sqrt{s}=r, \text{ or } p^2s=r^2.$$

Again, if $\alpha\beta+\gamma\delta=0$, we have

$$\alpha\beta=-\gamma\delta=\sqrt{-s},$$
$$\sqrt{-s}(\gamma+\delta-\alpha-\beta)=-r,$$

and
$$\alpha+\beta+\gamma+\delta=-p.$$

The last two equations give

$$2(\alpha+\beta)=-p+\frac{r}{\sqrt{-s}},\quad 2(\gamma+\delta)=-p-\frac{r}{\sqrt{-s}}.$$

Hence
$$4q=4(\alpha+\beta)(\gamma+\delta)=p^2+\frac{r^2}{s},$$

or
$$p^2s+r^2-4qs=0.$$

21. We have

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a},$$
$$\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta)=\frac{c}{a},$$
$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-\frac{d}{a}.$$

Hence, if $\alpha+\beta=\gamma+\delta$, we shall have

$$\alpha+\beta=-\frac{b}{2a}=\gamma+\delta;$$

$$\alpha\beta+\gamma\delta=\frac{2d}{b},$$

$$\alpha\beta+\gamma\delta+\frac{b^2}{4a^2}=\frac{c}{a}.$$

Hence
$$\frac{2d}{b}+\frac{b^2}{4a^2}=\frac{c}{a}.$$

22. We have

$$\{p_n+p_{n-3}x^3+p_{n-6}x^6+\ldots\}+x\{p_{n-1}+p_{n-4}x^3+p_{n-7}x^6+\ldots\}$$
$$+x^2\{p_{n-2}+p_{n-5}x^3+\ldots\}=0.$$

Hence

$$\{p_n+p_{n-3}x^3+\ldots\}^3+x^3\{p_{n-1}+p_{n-4}x^3+\ldots\}^3+x^6\{p_{n-2}+p_{n-5}x^3+\ldots\}^3$$
$$-3x^3\{p_n+p_{n-3}x^3+\ldots\}\{p_{n-1}+p_{n-4}x^3+\ldots\}\{p_{n-2}+p_{n-5}x^3+\ldots\}=0.$$

Now put $x^3=1-y$; then the term of highest degree in y will be y^n, and the term independent of y will be $A^3+B^3+C^3-3ABC$.

XLV. Pages 580—581.

1. (i)
$$f(x)=4x^3-12x^2-15x-4,$$
$$f'(x)=3(4x^2-8x-5).$$

The factor $2x+1$ is common to $f(x)$ and $f'(x)$.

Thus two of the roots of $f(x)=0$ are $-\frac{1}{2}$, $-\frac{1}{2}$, and the remaining root is

$$3-\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)=4.$$

(ii)
$$f(x)=x^4-6x^3+13x^2-24x+36,$$
$$f'(x)=2(2x^3-9x^2+13x-12).$$

The factor $x-3$ is common to $f(x)$ and $f'(x)$.

Thus two of the roots of $f(x)=0$ are 3, 3.

Hence $f(x)=0$ may be written

$$(x-3)^2(x^2+4)=0.$$

(iii)
$$f(x)=16x^4-24x^2+16x-3,$$
$$f'(x)=16(4x^3-3x+1).$$

The common factor of $f(x)$ and $f'(x)$ is $4x^2-4x+1=(2x-1)^2$, giving the roots $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$. The remaining root is $-\frac{3}{2}$, since the sum is zero.

(iv) $$f(x) = 2x^4 - 23x^3 + 84x^2 - 80x - 64,$$
$$f'(x) = 8x^3 - 69x^2 + 168x - 80.$$

The common factor of $f(x)$ and $f'(x)$ is $x^2 - 8x + 16 = (x-4)^2$, giving the roots 4, 4, 4. The remaining root of $f(x) = 0$ is $\frac{23}{2} - 4 - 4 - 4 = -\frac{1}{2}$.

2. The necessary and sufficient condition that the equation may have two equal roots is that

$$ax^3 + 3bx^2 + 3cx + d,$$

and

$$ax^2 + 2bx + c$$

may have a common factor.

Multiply the second expression by x and subtract from the first; then the given factor must be a factor of

$$bx^2 + 2cx + d.$$

Hence, if α be the root common to $f(x) = 0$ and $f'(x) = 0$, we have

$$a\alpha^2 + 2b\alpha + c = 0,$$

and

$$b\alpha^2 + 2c\alpha + d = 0.$$

Hence

$$\frac{\alpha^2}{2(bd - c^2)} = \frac{\alpha}{bc - ad} = \frac{1}{2(ac - b^2)},$$

whence the condition required is

$$(bc - ad)^2 = 4(ac - b^2)(bd - c^2).$$

3. By the previous question the common root α is equal to

$$\frac{1}{2}\frac{bc - ad}{ac - b^2}.$$

4. We may, without loss of generality, suppose that a', b', c', &c. are in ascending order of magnitude.

The equation is equivalent to

$$\lambda(x - a')(x - b') \ldots (x - k') - \Sigma a^2 \{(x - b')(x - c') \ldots (x - k')\} = 0.$$

If we suppose that there are n quantities $a', b', \ldots, k'$, the substitutions $-\infty$, $a', b', \ldots, k', +\infty$ in the expression on the left will give results whose signs are respectively

$$\lambda(-1)^n,\ (-1)^n,\ (-1)^{n-1},\ (-1)^{n-2}, \ldots\ldots,\ -1,\ \lambda.$$

Thus, whether λ is positive or negative, there are n changes of sign.

Hence there are n real roots.

5. The equation is equivalent to

$$(m + n^2 x)(x - \alpha)(x - \beta)(x - \gamma) \ldots - \Sigma \{a^2 (x - \beta)(x - \gamma) \ldots\} = 0.$$

We may without loss of generality suppose that α, β, γ, ... are in increasing order of magnitude.

If there are n of the quantities $\alpha, \beta, \gamma, \ldots$, and we substitute $-\infty$, α, $\beta, \ldots, +\infty$ in the left-hand member of the equation last written, the signs of the results will be respectively

$$(-1)^{n+1},\ (-1)^n,\ (-1)^{n-1}, \ldots, -1, +.$$

Hence there are $n+1$ changes of sign, whence it follows that the equation has $n+1$ real roots.

6. If we substitute $+\infty$, a_1, a_2, a_3, ..., a_{2n}, $-\infty$ in the left-hand member of the equation, the signs of the results will be

$$+, +b, -, -b, +, +b, \ldots, +, (-1)^n.$$

Hence, as b is positive, there are n real roots which lie respectively between a_1 and a_2, a_3 and a_4, &c.

7. The equation is equivalent to

$$x(x-b)(x-c)+x(x-c)(x-a)+x(x-a)(x-b)$$
$$+(x+d)(x-a)(x-b)(x-c)=0.$$

On the supposition that a, b, c are in descending order of magnitude, the substitutions $+\infty$, a, b, c, 0, $-\infty$ in the left-hand member of the equation, will give results whose signs are

$$+, +, -, +, -, +.$$

Hence there are four *real* roots.

Put $x=a-y$; then we have

$$(a-y)(a-b-y)(a-c-y)-(a-y)(a-c-y)y$$
$$-(a-y)(a-b-y)y-y(a-b-y)(a-c-y)(a+d-y)=0,$$

whence
$$y_1y_2y_3y_4=a(a-b)(a-c);$$
$$\therefore\ (a-\alpha)(a-\beta)(a-\gamma)(a-\delta)=a(a-b)(a-c).$$

Hence
$$\Sigma\frac{a^2}{(a-\alpha)(a-\beta)(a-\gamma)(a-\delta)}=\Sigma\frac{a}{(a-b)(a-c)}=0.$$

8. The equation will be obtained by rationalizing the equation

$$x=p\omega+q\omega^{-1}.$$

Now, since $\omega^5=1$

$$x^5=p^5+q^5+5(p^4q\omega^3+pq^4\omega^{-3})+10(p^3q^2\omega+p^2q^3\omega^{-1}).$$

Hence
$$x^5-p^5-q^5-10p^2q^2x=5pq(p^3\omega^3+q^3\omega^{-3})$$
$$=5pq(p\omega+q\omega^{-1})\{(p\omega+q\omega^{-1})^2-3pq\}$$
$$=5pqx(x^2-3pq).$$

Hence the equation required is

$$x^5-5pqx^3+5p^2q^2x-p^5-q^5=0.$$

9. We have

$$\overline{\alpha+\beta}+\overline{\gamma+\delta}=-4p \quad \text{(i)},$$
$$\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta)=6q \quad \text{(ii)},$$
$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-4r \quad \text{(iii)},$$
$$\alpha\beta\gamma\delta=s \quad \text{(iv)}.$$

Put $\alpha\beta+\gamma\delta=y.$

From (i) and (iii)

$$(\alpha+\beta)(\alpha\beta-\gamma\delta)=-4p\alpha\beta+4r$$
$$(\gamma+\delta)(\gamma\delta-\alpha\beta)=-4p\gamma\delta+4r.$$

Hence $-(\alpha+\beta)(\gamma+\delta)(\alpha\beta-\gamma\delta)^2=16\{p^2\alpha\beta\gamma\delta-pr(\alpha\beta+\gamma\delta)+r^2\};$

$$\therefore\ -(\alpha+\beta)(\gamma+\delta)(y^2-4s)=16(p^2s-pry+r^2).$$

Hence, from (ii),

$$y-16(p^2s-pry+r^2)/(y^2-4s)=6q,$$

or $$y^3-6qy^2-4(s-4pr)y+24sq-16p^2s-16r^2=0.$$

Again, put $z=(\alpha+\beta)(\gamma+\delta)$, then $y=6q-z$. Substitute this value in the equation for y and we shall obtain the second result, which could also be obtained independently in the same manner that the equation in y was found.

10. Put $$z=(\alpha+\beta-\gamma-\delta)^2$$
$$=\Sigma a^2+4\alpha\beta+4\gamma\delta-2\Sigma\alpha\beta$$
$$=16p^2-24q+4(\alpha\beta+\gamma\delta).$$

Hence $$\frac{1}{4}z-4p^2+6q=\alpha\beta+\gamma\delta.$$

The required result will therefore be obtained by substituting $\frac{1}{4}z-4p^2+6q$ for y in the equation for y found in the previous question.

11. $$\Sigma a_1{}^3a_2{}^2=\Sigma a_1a_2\,.\,\Sigma a_1{}^2a_2-2\Sigma a_1{}^3a_2a_3-2\Sigma a_1{}^2a_2{}^2a_3$$
$$=\Sigma a_1a_2\{\Sigma a_1a_2\,.\,\Sigma a_1-3a_1a_2a_3\}$$
$$-2a_1a_2a_3\{(\Sigma a_1)^2-2\Sigma a_1a_2\}-2a_1a_2a_3\Sigma a_1a_2$$
$$=p_2(-p_1p_2+3p_3)+2p_3(p_1{}^2-2p_2)+2p_3p_2$$
$$=p_2p_3-p_1p_2{}^2+2p_1{}^2p_3.$$

XLVI. Pages 596—598.

1. **(i)** $x^3-12x+65=0.$

Compare with $x^3-3abx+a^3+b^3=0;$

then $ab=4,\ a^3+b^3=65.$

Hence $a^6-65a^3+64=0;$

$$\therefore\ a^3=1 \text{ or } a^3=64.$$

Thus $a=1,\ b=4.$

Hence the roots required are

$$-(1+4),\ -(\omega+4\omega^2),\ -(\omega^2+4\omega).$$

(ii) $x^3-9x+28=0.$

Here $ab=3,\ a^3+b^3=28,$

whence $a=1,\ b=3.$

Hence the roots are $-(1+3),\ -(\omega+3\omega^2),\ -(\omega^2+3\omega).$

(iii) $x^3 - 48x - 520 = 0.$

Here $ab = 16,\ a^3 + b^3 = -520.$

Hence $a^6 + 520a^3 + 16^3 = 0,$ whence

$a^3 = -8$ or $a^3 = -512.$

Thus $a = -2,\ b = -8.$

Hence the roots are

$2 + 8,\ 2\omega + 8\omega^2,\ 2\omega^2 + 8\omega.$

(iv) $x^3 - 21x - 344 = 0.$

Here $ab = 7,\ a^3 + b^3 = -344.$

Hence $a = -1,\ b = -7.$

Thus the roots are

$(1 + 7),\ (\omega + 7\omega^2),\ (\omega^2 + 7\omega).$

(v) $x^3 - 2x + 5 = 0.$

Here $ab = \frac{2}{3},\ a^3 + b^3 = 5,$

whence $a^3 = \frac{5}{2} \pm \frac{1}{18}\sqrt{1929} = 4{\cdot}94002\ldots$ or $\cdot 05998\ldots.$

Hence $a = 1{\cdot}70\ldots,\ b = {\cdot}39\ldots.$

Thus the roots are

$-(1{\cdot}70\ldots + {\cdot}39\ldots),\ -(1{\cdot}70\ldots\omega + {\cdot}39\ldots\omega^2),$

and $-(1{\cdot}70\ldots\omega^2 + {\cdot}39\ldots\omega).$

[This method is of course much more laborious than Horner's process.]

(vi) $x^3 - 6x - 11 = 0.$

Here $ab = 2,\ a^3 + b^3 = -11,$

whence $a^3 = \frac{1}{2}\{-11 \pm \sqrt{89}\},$ &c.

2. (i) $x^4 + 2x^3 + 14x + 15 = 0.$

Add $(ax + \beta)^2$ to both sides; then

$x^4 + 2x^3 + a^2x^2 + (14 + 2a\beta)\,x + \beta^2 + 15 = (ax + \beta)^2.$

The left side is a perfect square, namely $(x^2 + x + \lambda)^2$, provided

$\lambda^2 = \beta^2 + 15,\ 2\lambda = 2a\beta + 14$ and $2\lambda + 1 = a^2.$

Hence $(\lambda^2 - 15)(2\lambda + 1) = (\lambda - 7)^2,$

or $\lambda^3 - 8\lambda - 32 = 0,$

whence by inspection $\lambda = 4$, corresponding values of a, β being 3, -1 respectively.

The original equation is therefore equivalent to

$(x^2 + x + 4)^2 = (3x - 1)^2.$

(ii) $x^4 - 12x - 5 = 0.$

Hence $x^4 - 12x - 5 + (ax + \beta)^2 = (ax + \beta)^2,$

or $x^4 + a^2x^2 + 2(a\beta - 6)\,x + \beta^2 - 5 = (ax + \beta)^2.$

The left side is the square of $x^2+\lambda$, provided

$$2\lambda=\alpha^2,\ \alpha\beta-6=0 \text{ and } \lambda^2=\beta^2-5.$$

Hence $2\lambda(\lambda^2+5)=36$, whence by inspection $\lambda=2$, corresponding values of α, β being 2 and 3 respectively. Thus the given equation is equivalent to

$$(x^2+2)^2=(2x+3)^2.$$

(iii) $$x^4-12x^2+24x+140=0.$$

Hence $$x^4-12x^2+24x+140+(\alpha x+\beta)^2=(\alpha x+\beta)^2.$$

The left side is the square of $x^2+\lambda$, provided

$$2\lambda=\alpha^2-12,\ 24+2\alpha\beta=0,\ \lambda^2=\beta^2+140;$$

$$\therefore (\lambda^2-140)(2\lambda+12)=144,$$

or $$\lambda^3+6\lambda^2-140\lambda-912=0.$$

By inspection $\lambda=12$, corresponding values of α, β being 6, -2.

Thus $$(x^2+12)^2=(6x-2)^2.$$

(iv) $$4x^4+4x^3-7x^2-4x-12=0.$$

Hence $$4x^4+4x^3-7x^2-4x-12+(\alpha x+\beta)^2=(\alpha x+\beta)^2.$$

The left side is the square of $2x^2+x+\lambda$, provided

$$4\lambda+1=-7+\alpha^2,$$

$$2\lambda\ \ =-4+2\alpha\beta,$$

and $$\lambda^2\ \ =-12+\beta^2.$$

Hence $$(\lambda^2+12)(4\lambda+8)=(\lambda+2)^2,$$

or $$4\lambda^3+7\lambda^2+44\lambda+92=0,$$

whence by inspection $\lambda=-2$, corresponding values of α, β being 0, 4.

Thus we have $$(2x^2+x-2)^2=16.$$

3. (i)

$$f(x)=x^3-3x+6,$$
$$f_1(x)=3(x^2-1),$$
$$f_2(x)=2(x-3),$$
$$f_3(x)=-8.$$

Hence the signs corresponding to different values of x are as under:

$$\begin{matrix} +\infty\ ,\ +,\ +,\ +,\ - \\ 0\ \ ,\ +,\ -,\ -,\ - \\ -2\ ,\ +,\ +,\ -,\ - \\ -3\ ,\ -,\ +,\ -,\ - \\ -\infty\ ,\ -,\ +,\ -,\ - \end{matrix}$$

Thus there is one real root between -2 and -3.

(ii)

$$f(x)=x^3-x^2-33x+61,$$
$$f'(x)=3x^2-2x-33,$$
$$f_2(x)=50x-129,$$
$$f_3(x)=+.$$

The signs corresponding to different values of x are as under:

$$\begin{array}{l} +\infty\ ,\ +,\ +,\ +,\ + \\ \quad 6\ ,\ +,\ +,\ +,\ + \\ \quad 5\ ,\ -,\ +,\ +,\ + \\ \quad 2\ ,\ -,\ -,\ -,\ + \\ \quad 1\ ,\ +,\ -,\ -,\ + \\ -6\ ,\ +,\ +,\ -,\ + \\ -7\ ,\ -,\ +,\ -,\ + \end{array}$$

Thus the roots lie between -7 and -6, 1 and 2, 5 and 6.

(iii)
$$\begin{aligned} f(x) &= 2x^4 - x^2 - 10x + 3, \\ f'(x) &= 2\,(4x^3 - x - 5), \\ f_2(x) &= x^2 + 15x - 6, \\ f_3(x) &= -185x + 73, \\ f_4(x) &= -. \end{aligned}$$

The signs corresponding to different values of x are

$$\begin{array}{l} +\infty\ ,\ +,\ +,\ +,\ -,\ - \\ \quad 2\ ,\ +,\ +,\ +,\ -,\ - \\ \quad 1\ ,\ -,\ -,\ +,\ -,\ - \\ \quad 0\ ,\ +,\ -,\ -,\ +,\ - \\ -\infty\ ,\ +,\ -,\ +,\ +,\ - \end{array}$$

Thus there are two real roots, one between 2 and 1 and the other between 1 and 0.

(iv)
$$\begin{aligned} f(x) &= x^4 - 14x^2 + 16x + 9, \\ f'(x) &= 4\,(x^3 - 7x + 4), \\ f_2(x) &= 7x^2 - 12x - 9, \\ f_3(x) &= 17x - 38, \\ f_4(x) &= +. \end{aligned}$$

The signs corresponding to different values of x are

$$\begin{array}{l} +\infty\ ,\ +,\ +,\ +,\ +,\ + \\ \quad 3\ ,\ +,\ +,\ +,\ +,\ + \\ \quad 2\ ,\ +,\ -,\ -,\ 0\ ,\ + \\ \quad 0\ ,\ +,\ +,\ -,\ -,\ + \\ -1\ ,\ -,\ +,\ +,\ -,\ + \\ -4\ ,\ -,\ -,\ +,\ -,\ + \\ -5\ ,\ +,\ -,\ +,\ -,\ + \end{array}$$

Thus there are two real roots between 3 and 2, one between 0 and -1, and one between -4 and -5.

(v)
$$f(x)=x^4-7x^2+3x-20,$$
$$f'(x)=4x^3-14x+3,$$
$$f_2(x)=14x^2-9x+80.$$

Since the roots of $f_2(x)=0$ are imaginary it can never change sign. It is therefore unnecessary to find the other functions, for $f_4(x)$ has a constant sign and $f_3(x)$ cannot change sign except it vanishes, and for that value $f_2(x)$ and $f_4(x)$ have opposite signs. Thus $f_4(x)$ must be *negative*, (since $f_2(x)$ is always positive), and therefore $f_2(x)$, $f_3(x)$, $f_4(x)$ will always give one and only one change of sign.

By using only the three functions $f(x)$, $f'(x)$ and $f_2(x)$ we find that there are two real roots which lie between 2 and 3 and between -3 and -4 respectively.

4.
$$f(x)=x^3+3px^2+3qx+r,$$
$$f'(x)=3(x^2+2px+q).$$
$$f_2(x)=2(p^2-q)x+pq-r,$$
$$f_3(x)=\{6pqr+3p^2q^2-4p^3r-4q^3-r^2\}/4(p^2-q)^2.$$

The signs for the values $-\infty$, $+\infty$ of x are, if $p^2<q$,

$$-\infty,\ -,\ +,\ +,\ \pm$$
$$+\infty,\ +,\ +,\ -,\ \pm$$

Thus, whether the sign of $f_3(x)$ be $+$ or $-$, there can only be *one* real root.

5.
$$f(x)=x^3+px^2+r,$$
$$f'(x)=3x^2+2px,$$
$$f_2(x)=2p^2x-9r,$$
$$f_3(x)=-\frac{9}{2}r(4p^3+27r)/p^2.$$

The four roots will all be real if $f_3(x)$ be positive; that is, if $4p^3r+27r^2<0$. For the signs of Sturm's Functions will then be

$$-\infty,\ -,\ +,\ -,\ +$$
$$+\infty,\ +,\ +,\ +,\ +.$$

6. Let
$$f(x)=a_0x^n+a_1x^{n-1}+\ldots a_{n-1}x+a_n.$$
Then
$$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0.$$

If x be an even integer, $a_0x^n+\ldots+a_{n-1}x$ must be even, and therefore $(a_0x^n+\ldots+a_{n-1}x)+a_n$ cannot be zero since a_n is odd.

Again, if x be an odd integer, $2k+1$ suppose,

$$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$$
$$=a_0(x^n-1)+a_1(x^{n-1}-1)+\ldots a_{n-1}(x-1)+(a_0+a_1+a_n+\ldots+a_{n-1}+a_n)$$
$$=\text{an even integer}+(a_0+a_1+\ldots+a_n),$$

and the sum of an even integer and an odd integer cannot be zero.

7.

1	0	−2	−5	(2·0945514
	2	2	−1000000	
	4	100000		
	600			
	609	105481	−50671000	
	618	11104300		
	6270			
	6274	11129396	− 6153416	
	6278	11154508		
	62782	1115760	− 574616	
		1116076	− 16581	
			− 5421	
			− 957	
			− 69	

8. (i) By substitution we see that $f(1)$ is +, $f(\frac{3}{2})$ is −, and $f(2)$ is +. Hence, as there can be only two positive, they lie between 1 and $1\frac{1}{2}$ and between $1\frac{1}{2}$ and 2 respectively.

1	0	−7	7	(1·3568...
	1	−6	1000	
	2	−400		
	30			
	33	−301	97000	
	36	−19300		
	390			
	395	−17325	10375	
	400	−15325		
	4005	−1508	1327	
			127	

1	0	−7	7	(1·6920...
	1	−6	1000	
	2	−400		
	30			
	36	−184	−104000	
	42	+ 6800		
	480			
	489	11201	− 3191	
	498	15685		
	5007	1578	− 35	

(ii) $x^3 - 8x - 40 = 0$.

By inspection the only real positive root is between 4 and 5.

```
1   0    -8      -40     (4·1891
    4     8    ÷ 8000
    8  | 4000
   120
   121   4121  | - 3879000
   122 | 424300
  1230
  1231  434148 |  -  405816
  1232 |
  1233  444004  -     6216
                -     1776
```

(iii) By inspection the roots are between 0 and 1, 1 and 2, 3 and 4 respectively.

```
1   -60    900    -3000    (·4679
    -56    676  | - 296000
   -52  | 46800
   -480
   -474  43956 |  -  32264
   -468 | 41148
   -462  4080    -    3662
                 -      17

1   -6      9      -3     (1·6527
    -5      4    | +1000
   -4     | 0
   -30
   -24    -144   |  136000
   -18  | -25200
   -120
   -115  -25775  |   7125
   -110 | -26325
   -105  -2634       1857
                       46

1   -6      9      -3     (3·87...
    -3      0    | -3000
     0    | 0
     30
     38    304  | -568000
     46  | 67200
     540
     547  71029 | -70797
     554 | 74907
     561
```

(iv) $x^4+x^3-4x^2-16=0.$

By inspection there is only one real root which lies between 2 and 3.

1	1	−4	0	−16	(2·2...
	3	2	4	− 80000	
	5	12	28000		
	7	2600			
	90				
	92	2784	33568	− 12864	
	94	2972	39512		
	96	3164			
	98				

(v) $x^4-14x^2+16x+9=0.$

By Sturm's Theorem it will be found that there are two roots between 2 and 3. [See solution of 3 (iv).]

9. (i) $f(x)=x^4+2x^3-23x^3-24x+144,$

$f'(x)=2(2x^3+3x^2-23x-12).$

If we now attempt to apply Sturm's Theorem we shall be led to find that $f(x)$ and $f'(x)$ have the common factor x^2+x-12.

Hence the given equation is

$$(x-3)^2(x+4)^2=0.$$

(ii) $f(x)=x^4-26x^3+48x+9,$

$f'(x)=4(x^3-13x+12).$

It will be found that $f(x)$ and $f'(x)$ have the common factor $x-3$. Hence $f(x)=(x-3)^2(x^2+6x+1)$.

Thus the roots of the given equation are 3, 3, $-3\pm\sqrt{8}$.

10. By Descartes' Rule of Signs the equation cannot have more than 3 positive roots, and cannot have more than 1 negative root.

The equation may be written

$$(x^6-7x^3+12\tfrac{1}{4})+(15x^2+3x-16\tfrac{1}{4})=0.$$

The first term of the left-hand member is always positive for real values of x, and the second term is positive if $x>1$. Hence there can be no real positive root greater than 1.

Change x into $-x$, then we have

$$x^6+7x^3+15x^2-3x-4=0,$$

the left-hand member of which is clearly positive when x is 1 or has any greater value. Hence real negative roots lie between 0 and -1.

11. The only possible commensurable roots are

$$\pm6,\ \pm3,\ \pm2,\ \pm1,\ \pm\frac{1}{2},\ \pm\frac{3}{2},$$

and of these it will be found on trial that 1, 3 and $-\frac{1}{2}$ satisfy the equation.

Thus the left-hand member of the equation is divisible by

$$(x-1)(x-3)(2x+1),$$

and hence is found to be equivalent to

$$(x-1)(x-3)(2x+1)(x^2+2)=0.$$

12. We have $x^2\{x^2-3(1-p)\}^2=(3px^2-1)^2$.

Hence the equation required is

$$y\{y-3(1-p)\}^2-(3py-1)^2=0,$$

i.e. $y^3-(9p^2-6p+6)y^2+(9p^2-12p+9)y-1=0.$

Since $9p^2-6p+6$ and $9p^2-12p+9$ are positive for all real values of p, it follows by Descartes' Rule of Signs that all the real roots of the cubic in y are positive.

Now, if p be negative, the signs of

$$f(\infty),\ f(3-3p),\ f(1),\ f(0)$$

are $+$, $-$, $+$, $-$ respectively. Hence in this case there are 3 real roots.

Again, if p be positive and greater than $\frac{1}{2}$, the substitutions $+\infty$, 1, $\frac{1}{3p}$, 0 (which are in descending order) give the signs $+$, $-$, $+$, $-$ respectively. Hence there are three positive roots in this case.

And, if p be positive and less than $\frac{1}{2}$, the substitutions $+\infty$, $3(1-p)$, 1, 0 (which are in descending order) give the signs

$$+,\ -,\ +,\ -$$

respectively. Hence in this case also the cubic in y has three real positive roots.

Since the cubic in y has three real *positive* roots for all values of p, it follows that the three roots of the original cubic must be real.

13. Substitute for x the values $+\infty$, 1, 0 in succession; then the signs of the left-hand members will be $+$, $-$, $+$ respectively. Thus there are always two, and therefore three, real roots.

We have $\alpha\beta\gamma=-1,\ \alpha+\beta+\gamma=3p,\ \beta\gamma+\gamma\alpha+\alpha\beta=3(p-1).$

Hence $$\beta\gamma+\gamma\alpha+\alpha\beta=\alpha+\beta+\gamma-3,$$

whence from the first equation,

$$\beta^2\gamma^2-\beta\gamma(\beta+\gamma-3)-\beta-\gamma+1=0,$$

i.e. $$(\beta\gamma-\beta+1)(\beta\gamma-\gamma+1)=0.$$

Whence, either $$\beta(1-\gamma)=1 \text{ or } \gamma(1-\beta)=1.$$

Then, if $$\beta\gamma-\beta+1=0,$$

$$\alpha\beta\gamma-\alpha\beta+\alpha=0;$$

$$\therefore\ \alpha\beta-\alpha+1=0,$$

and, from $\alpha\beta\gamma=-1$, $$\alpha\gamma-\gamma+1=0.$$

So also, if $\beta\gamma-\gamma+1=0$ we have

$$\alpha\beta-\beta+1=0 \text{ and } \alpha\gamma-\alpha+1=0.$$

14. Let the roots of the quintic be $\alpha, \beta, \gamma, \delta, \epsilon$, and let $y=\alpha+\beta$, the sum of any two roots. Also let $S_1=\gamma+\delta+\epsilon$, $S_2=\gamma\delta+\delta\epsilon+\epsilon\gamma$, $S_3=\gamma\delta\epsilon$.

Then we have

$$\begin{aligned} y+S_1 &= 0,\\ \alpha\beta+(\alpha+\beta)S_1+S_2 &= 0,\\ \alpha\beta S_1+(\alpha+\beta)S_2+S_3 &= 0,\\ \alpha\beta S_2+(\alpha+\beta)S_3 &= p,\\ \alpha\beta S_3 &= -q. \end{aligned}$$

Hence

$$\begin{aligned} \alpha\beta-y^2+S_2 &= 0,\\ -\alpha\beta y+yS_2+S_3 &= 0,\\ \alpha\beta\,.\,S_2+yS_3 &= p,\\ \alpha\beta\,.\,S_3 &= -q. \end{aligned}$$

Hence $S_2=y^2-\alpha\beta$, $S_3=2\alpha\beta y-y^3$;

and then

$$\begin{aligned} \alpha^2\beta^2-3\alpha\beta y^2+y^4+p &= 0,\\ 2\alpha^2\beta^2 y-\alpha\beta y^3+q &= 0. \end{aligned}$$

Hence $5y^3(y^7+py^3-3qy^2)=(2y^5+2py-q)^2$,

i.e. $y^{10}-3py^6-11qy^5-4p^2y^2+4pqy-q^2=0$.

15.

$$\begin{aligned} x^4+4ax^3+6a^2x^2+4ax+1 &= 0,\\ (x^2+x^{-2})+4a(x+x^{-1})+6a^2 &= 0, \end{aligned}$$

whence, if $x+x^{-1}=y$, we have

$$y^2+4ay+6a^2-2=0.$$

The values of y are imaginary unless $1-a^2$ is positive, and x is imaginary if y is imaginary. Hence $1>a^2$.

The values of y are given by

$$y=-2a\pm\sqrt{2-2a^2}.$$

Now $x+x^{-1}$ must, if x be real, be numerically greater than 2.

But, the condition that $-2a\pm\sqrt{2-2a^2}$ is numerically greater than 2 is that

$$a^2>\frac{1}{9}.$$

16. The condition that $\frac{1+\alpha}{1-\alpha}$ may be a root of the equation is that

$$(1+\alpha)^4+\alpha(1+\alpha)^3(1-\alpha)-6(1+\alpha)^2(1-\alpha)^2-\alpha(1+\alpha)(1-\alpha)^3+(1-\alpha)^4=0,$$

i.e. $-4(\alpha^4+a\alpha^3-6\alpha^2-a\alpha+1)=0$,

which is true since α is a root.

The equation is equivalent to that obtained by changing x into $-\frac{1}{x}$; and from this it follows that α, $-\frac{1}{\alpha}$, $\frac{1+\alpha}{1-\alpha}$, $\frac{\alpha-1}{\alpha+1}$ are all roots of the equation, and these are the only roots since the equation is of the fourth degree.

17. $\Sigma\alpha^2\beta^2\gamma = \Sigma\alpha\beta\gamma \,.\, \Sigma\alpha\beta - 3\Sigma\alpha^2\beta\gamma\delta - 10\Sigma\alpha\beta\gamma\delta\epsilon,$

$\Sigma\alpha^2\beta\gamma\delta = \Sigma\alpha \,.\, \Sigma\alpha\beta\gamma\delta - 5\Sigma\alpha\beta\gamma\delta\epsilon.$

Hence $\Sigma\alpha^2\beta^2\gamma = \Sigma\alpha\beta\gamma \,.\, \Sigma\alpha\beta - 3\Sigma\alpha \,.\, \Sigma\alpha\beta\gamma\delta + 5\Sigma\alpha\beta\gamma\delta\epsilon$

$= -p_3p_2 + 3p_1p_4 - 5p_5.$

18. We have

$$\Sigma a_1 = 0, \quad \Sigma a_1a_2 = -5p, \quad \Sigma a_1a_2a_3 = 0,$$

$$\Sigma a_1a_2a_3a_4 = 5p^2, \quad a_1a_2a_3a_4a_5 = q.$$

Now $\Sigma a_1^4a_2^3a_3^2a_4 = \Sigma a_1a_2a_3a_4 \,.\, \Sigma a_1^3a_2^2a_3 - 3\Sigma a_1^4a_2^3a_3a_4a_5$

$- 2\Sigma a_1^4a_2^2a_3^2a_4a_5 - \Sigma a_1^3a_2^3a_3^2a_4a_5.$

$\Sigma a_1a_2a_3a_4 \,.\, \Sigma a_1^3a_2^2a_3 = 5p^2\{\Sigma a_1a_2a_3 \,.\, \Sigma a_1^2a_2 - 3\Sigma a_1^3a_2a_3a_4 - 2\Sigma a_1^2a_2^2a_3a_4\}$

$= -5p^2\{3\Sigma a_1a_2a_3a_4 \,.\, \Sigma a_1^2 - 3\Sigma a_1^2a_2a_3a_4a_5$

$+ 2\Sigma a_1a_2a_3a_4 \,.\, \Sigma a_1a_2 - 8\Sigma a_1^2a_2a_3a_4a_5\}$

$= -5p^2\{15p^2 \,.\, 10p + 10p^2(-5p)$

$- 11a_1a_2a_3a_4a_5 \,.\, \Sigma a_1\} = -500p^5.$

Also $3\Sigma a_1^4a_2^3a_3a_4a_5 + 2\Sigma a_1^4a_2^2a_3^2a_4a_5 + \Sigma a_1^3a_2^3a_3^2a_4a_5$

$= a_1a_2a_3a_4a_5(3\Sigma a_1^3a_2^2 + 2\Sigma a_1^3a_2a_3 + \Sigma a_1^2a_2^2a_3)$

$= q\,(3\Sigma a_1a_2 \,.\, \Sigma a_1 - 4\Sigma a_1^3a_2a_3 - 5\Sigma a_1^2a_2^2a_3 - 9\Sigma a_1^2a_2a_3a_4)$

$= q\{-15p\,(\Sigma a_1 \,.\, \Sigma a_1a_2 - 3\Sigma a_1a_2a_3) - 4\Sigma a_1a_2a_3 \,.\, \Sigma a_1^2$

$- 5\,(\Sigma a_1a_2a_3 \,.\, \Sigma a_1a_2 - a_1a_2a_3a_4)\}$

$= 5q^2.$

Hence $\Sigma a_1^4a_2^3a_3^2a_4 = -500p^5 - 5q^2.$

MISCELLANEOUS EXAMPLES.

Pages 599—617.

1. (i) It is at once seen that $b-c$, $c-a$ and $a-b$ are factors. Hence as the expression is symmetrical and of the fifth degree it must be equal to

$$(b-c)(c-a)(a-b)\{L(a^2+b^2+c^2)+M(bc+ca+ab)\},$$

where L and M are numerical.

Equating the coefficients of a^4b and a^3b^2 in the two expressions, we have

$$-L=1, \text{ and } L-M=1.$$

(ii) $a^3bcd-a^2(b^2c^2+c^2d^2+d^2b^2)+abcd(b^2+c^2+d^2)-b^2c^2d^2$
$=(ab-cd)(ac-bd)(ad-bc).$

(iii) $2a^3+a^2(b+c)+a(b^2+c^2-3bc)+(b+c)(2b^2+2c^2-bc)$
$=(a+b+c)(2a^2+2b^2+2c^2-bc-ca-ab).$

(iv) $a^4(b+c)+a^3(b+c)^2+a^2\{b^3+c^3+2bc(b+c)\}$
$+a\{b^4+c^4+2b^3c+2bc^3+2b^2c^2\}+b^4c+b^2c^2+b^2c^3+bc^4$
$=(b+c)\{a^4+a^3(b+c)+a^2(b^2+bc+c^2)+a(b^3+b^2c+bc^2+c^3)$
$+b^3c+bc^3\}$
$=(b+c)\{b^3(c+a)+b^2a(c+a)+b(c^3+c^2a+ca^2+a^3)$
$+a(c^3+c^2a+ca^2+a^3)\}$
$=(b+c)(c+a)\{b^3+b^2a+b(c^2+a^2)+a(c^2+a^2)\}$
$=(b+c)(c+a)(a+b)(a^2+b^2+c^2).$

2. The given expression on the left

$$=\Sigma\frac{(b\gamma-c\beta)(a+\alpha)}{a^2-\alpha^2}$$

$$=\frac{1}{a^2-\alpha^2}\{(b\gamma-c\beta)(a+\alpha)+(c\alpha-a\gamma)(b+\beta)+(a\beta-b\alpha)(c+\gamma)\}=0.$$

3. $a^2+x^2=yz+zx+xy+x^2=(x+y)(x+z)$, &c.

Hence $\Sigma\dfrac{1}{yz(a^2+x^2)}=\dfrac{\Sigma x(y+z)}{xyz(y+z)(z+x)(x+y)}$

$$=\frac{2a^2}{xyz\sqrt{\{(a^2+x^2)(a^2+y^2)(a^2+z^2)\}}}.$$

4. $x^2=x^2+yz+zx+xy=(x+y)(x+z)$, &c.

Hence $(y+z)^2(z+x)^2(x+y)^2+2x^2y^2z^2=3(y+z)^2(z+x)^2(x+y)^2$.

Also $x^4(y+z)^2=(y+z)^2(z+x)^2(x+y)^2=y^4(z+x)^2=z^4(x+y)^2$.

Hence $\Sigma x^4(y+z)^2=3(y+z)^2(z+x)^2(x+y)^2$.

5. Let the quotient be $\phi(x)$ and the remainder $Ax+B$.

Then $$f(x)=(x-a)(x-b)\phi(x)+Ax+B.$$

In this identity put $x=a$ and $x=b$ successively; then

$$f(a)=Aa+B,$$

and $$f(b)=Ab+B.$$

Hence $$A=\frac{f(a)-f(b)}{a-b} \text{ and } B=\frac{af(b)-bf(a)}{a-b}.$$

6. Since $$(b-c)+(c-a)+(a-b)=0,$$

$$\Sigma(b-c)^2=-2\Sigma(b-c)(c-a).$$

Square both sides; then $\Sigma(b-c)^4+2\Sigma(b-c)^2(c-a)^2$

$$=4\Sigma(b-c)^2(c-a)^2+8(b-c)(c-a)(a-b)\Sigma(b-c);$$

$$\therefore \Sigma(b-c)^4=2\Sigma(b-c)^2(c-a)^2 \quad \text{.....(i).}$$

Again $$\{\Sigma(b-c)^2\}^2=4\Sigma(b-c)^2(c-a)^2+8(b-c)(c-a)(a-b)\Sigma(b-c)$$

$$=4\Sigma(b-c)^2(c-a)^2 \quad \text{..... (ii).}$$

Hence $$2\Sigma(b-c)^4=\{\Sigma(b-c)^2\}^2.$$

7. Put $b-c=\alpha$, $c-a=\beta$, $a-b=\gamma$; then $\alpha+\beta+\gamma=0$. Hence α, β, γ are the roots of the cubic,

$$x^3+px+q=0,$$

provided that $p=\Sigma\beta\gamma$ and $q=-\alpha\beta\gamma$.

Then $$\Sigma\alpha^6=3q^2-2p^3, \quad \text{[Art. 129]},$$

$$a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}\Sigma\alpha^2=-p.$$

Hence $$3\Sigma\alpha^6-\frac{3}{4}(\Sigma\alpha^2)^3=9q^2-6p^3+6p^3=9q^2.$$

Hence the square root required $=\pm 3q=\pm 3\alpha\beta\gamma=\pm 3(b-c)(c-a)(a-b)$.

8. Put $b-c=\alpha$, $c-a=\beta$, $a-b=\gamma$.

Then, as in Art. 129,

$$\Sigma\alpha^7=-7p^2q, \quad \Sigma\alpha^4=2p^2.$$

Hence $$\Sigma\alpha^4=-\frac{7}{2}\Sigma\alpha^4 \cdot q=\frac{7}{2}\alpha\beta\gamma\Sigma\alpha^4.$$

9. $\Sigma bc(a-b)(a-c)=\Sigma b^2c^2-abc\Sigma a=\frac{1}{2}\Sigma(ab-ac)^2$, which is necessarily positive.

10. Since $a+b+c=0$,

$$(c-a)(a-b)=-a^2+a(b+c)-bc=2a(b+c)-bc=2\Sigma bc-3bc.$$

Hence $$a\{(c-a)(a-b)\}^2=a(2\Sigma bc-3bc)^2$$

$$=4a(\Sigma bc)^2-12abc\Sigma bc+9abc \cdot bc.$$

Hence $\Sigma a(c-a)^2(a-b)^2$

$$=4\Sigma a(\Sigma bc)^2-36abc\Sigma bc+9abc\Sigma bc$$

$$=-27abc\Sigma bc.$$

11. Since $a+b+c+d=0$, a, b, c, d will for some values of p, q, r be the roots of

$$x^4+px^2+qx+r=0.$$

Hence $\Sigma a=0$, $\Sigma ab=p$, $\Sigma abc=-q$ and $abcd=r$.

$$\Sigma a^2=(\Sigma a)^2-2\Sigma ab=-2p.$$

$$\Sigma a^3=\Sigma a^2\,.\,\Sigma a-\Sigma a^2b=-\Sigma a^2b=-\Sigma ab\,.\,\Sigma a+3\Sigma abc=-3q.$$

$$\Sigma a^4+p\Sigma a^2+q\Sigma a+4r=0;$$

$$\therefore\ \Sigma a^4-2p^2+4r=0.$$

$$\Sigma a^5+p\Sigma a^3+q\Sigma a^2+r\Sigma a=0;$$

$$\therefore\ \Sigma a^5-3pq-2pq=0.$$

Hence $$\frac{1}{5}\Sigma a^5=pq=\left(-\frac{1}{2}\Sigma a^2\right)\left(-\frac{1}{3}\Sigma a^3\right).$$

Also $$(\Sigma a^2)^2=4p^2=2\Sigma a^4+8r=2\Sigma a^4+8abcd.$$

Again $$x^4+px^2+qx+r=(x-a)(x-b)(x-c)(x-d).$$

Hence $$\log(1+px^{-2}+qx^{-3}+rx^{-4})=\log(1-ax^{-1})+\&c.$$

The coefficient of $x^{-(2n+1)}$ on the right is $-\frac{1}{2n+1}\Sigma a^{2n+1}$. But it is obvious that there can be no term in the expansion of the expression on the left containing an *odd* power of x^{-1} which does not contain q as a factor.

Hence Σa^{2n+1} is divisible by q, and $3q=-\Sigma a^3$.

12. Since $\Sigma a=0=\Sigma a^2$, it follows that $\Sigma ab=0$. Hence a, b, c, d can be taken to be the roots of the equation $x^4+px+q=0$.

Substitute a, b, c, d in succession and add: then

$$\Sigma a^4+p\Sigma a+4q=0;$$

$$\therefore\ \Sigma a^4=-4q.$$

Also, $$\Sigma a^8+p\Sigma a^5+q\Sigma a^4=0,$$

and $$\Sigma a^5+p\Sigma a^2+q\Sigma a=0.$$

Hence $\Sigma a^5=0$ and $\Sigma a^8=-q\Sigma a^4=\frac{1}{4}(\Sigma a^4)^2$.

13. Put $\lambda=(a+b+c)=-(d+e+f)$.

Then $$(a+b)(b+c)(c+a)+(d+e)(e+f)(f+d)$$

$$=(\lambda-c)(\lambda-a)(\lambda-b)-(\lambda+f)(\lambda+d)(\lambda+e)$$

$$=\lambda^3-(a+b+c)\lambda^2+(bc+ca+ab)\lambda-abc$$

$$-\{\lambda^3+(d+e+f)\lambda^2+(de+ef+fd)\lambda+def\}$$

$$=-(bc+ca+ab)(d+e+f)-(de+ef+fd)(a+b+c)-abc-def$$

$$=-\Sigma abc.$$

Now, since $\Sigma a=0$, $\Sigma a(\Sigma a^2-\Sigma bc)=0$, that is $\Sigma a^3-3\Sigma abc=0$. Hence, as $\Sigma a^3=0$, it follows that $\Sigma abc=0$.

Again, since $\Sigma a=0$, and $\Sigma abc=0$, we can put

$$x^6+px^4+rx^2+sx+t=(x-a)(x-b)(x-c)(x-d)(x-e)(x-f).$$

Hence
$$\log\left(1+\frac{p}{x^2}+\frac{r}{x^4}+\frac{s}{x^5}+\frac{t}{x^6}\right)$$
$$=\log\left(1-\frac{a}{x}\right)+\log\left(1-\frac{b}{x}\right)+\ldots\ldots$$

Equate coefficients of x^{-7}; then $\frac{1}{7}\Sigma a^7$ is coefficient of x^{-7} in

$$+\frac{1}{2}\left\{\frac{p}{x^2}+\frac{r}{x^4}+\frac{s}{x^5}+\frac{t}{x^6}\right\}^2$$
$$-\frac{1}{3}\left\{\frac{p}{x^2}+\frac{r}{x^4}+\ldots\ldots\right\}^3.$$

Hence
$$\frac{1}{7}\Sigma a^7=ps.$$

Equate coefficients of x^{-5}; then

$$\frac{1}{5}\Sigma a^5=-s.$$

Also
$$\Sigma ab=p \text{ and } \Sigma a=0;$$
$$\therefore\ \frac{1}{2}\Sigma a^2=-p.$$

Hence
$$\frac{1}{7}\Sigma a^7=\frac{1}{5}\Sigma a^5.\frac{1}{2}\Sigma a^2.$$

14. This follows at once from the identity

$$(ax+by+cz)^3+(by+cx+az)^3+(cz+ay+bx)^3$$
$$-3(ax+by+cz)(by+cx+az)(cz+ay+bx)$$
$$=(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz). \quad \text{[Art. 430, Ex. 1.]}$$

15. If ω be an imaginary cube root of unity, $1+\omega+\omega^2=0$.
Hence we have

$$u+v+w=3(x+y+z),$$
$$u+\omega v+\omega^2 w=-3a(x+\omega y+\omega^2 z),$$

and
$$u+\omega^2 v+\omega w=-3a(x+\omega^2 y+\omega z).$$

Hence $u^3+v^3+w^3-3uvw=(u+v+w)(u+\omega v+\omega^2 w)(u+\omega^2 v+\omega w)$
$$=27a^2(x^3+y^3+z^3-3xyz).$$

16. $X-A=(b-c)(y-z)$, &c.
Hence $(X-A)(X-B)(X-C)=(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$, whence it follows that

$$(X-A)(X-B)(X-C)=(Y-A)(Y-B)(Y-C)$$
$$=(Z-A)(Z-B)(Z-C).$$

It is at once seen that $b-c$, $c-a$, $a-b$, $y-z$, $z-x$, $x-y$ are all factors of $XYZ-ABC$. Moreover the coefficient of

$$y^2z=(a^2b+b^2c+c^2a)-(a^2c+b^2a+c^2b)=-(b-c)(c-a)(a-b).$$

Hence $XYZ-ABC=(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$.

17. We have to prove that

$$(ca'-c'a)^2+b^2a'c'+b'^2ac>bb'(ac'+a'c),$$

or, since $a'c'$ is positive from $4a'c'>b'^2$,

$$4a'c'(ca'-c'a)^2+4b'^2aa'cc'+\{2ba'c'-b'(ac'+a'c)\}^2-b'^2(ac'+a'c)^2>0,$$

or $$(4a'c'-b'^2)(ca'-c'a)^2+\{2ba'c'-b'(ac'+a'c)\}^2>0,$$

which is true since $4a'c'-b'^2>0$.

18. Since b and $abc+$&c. are both positive,

$$b\{abc+2lmn-al^2-bm^2-cn^2\}>0,$$

$$\text{i.e. } (bc-l^2)(ab-n^2)-(bm-ln)^2>0;$$

$$\therefore (bc-l^2)(ab-n^2)>0.$$

Hence, as $bc-l^2>0$, $\quad ab-n^2>0$.

Since $ab-n^2>0$ and b is positive, a must be positive.

Also, since $bc-l^2>0$ and b is positive, c must be positive.

Then $$c(abc+2lmn-al^2-bm^2-cn^2)>0,$$

$$\text{i.e. } (ca-m^2)(bc-l^2)-(cn-lm)^2>0,$$

whence it follows that $ca-m^2>0$.

19. Multiply the first by x and add to the second; then

$$a(b-c)x^2+a(b-c)yz+\{b(c-a)+c(a-b)\}(xy+xz)=0,$$

$$\text{i.e. } a(b-c)(x^2+yz-xy-xz)=0,$$

$$\text{or } a(b-c)(x-y)(x-z)=0.$$

Hence we must have

$$(x-y)(x-z)=0,\ (y-z)(y-x)=0 \text{ and } (z-x)(z-y)=0,$$

whence it follows that $x=y=z$.

20. Equate coefficients of x^2, &c. on the two sides of the given identity; then

$$ll'=a,\ mm'=b,\ nn'=c,$$

$$mn'+m'n=2f,\quad nl'+n'l=2g,\quad lm'+l'm=2h.$$

Hence $4(gh-af)=(lm'+l'm)(ln'+l'n)-2ll'(mn'+m'n)$

$$=(lm'-l'm)(ln'-l'n).$$

Hence $4(gh-af)(mn'-m'n)=-(mn'-m'n)(nl'-n'l)(lm'-l'm)=$&c.

21.
$$\left.\begin{aligned} cy+bz&=ax+by+cz\\ az+cx&=ax+by+cz\\ bx+ay&=ax+by+cz \end{aligned}\right\}.$$

From the first two equations, we have

$$\frac{x}{(b-c)(b+c-a)}=\frac{y}{(a-c)(c+a-b)}=\frac{z}{c(a+b-c)}.$$

Substitute in the third equation; then

$$(a-b)(b-c)(b+c-a)+(a-b)(c-a)(c+a-b)+c^2(a+b-c)=0,$$

i. e. $(b+c-a)(c+a-b)(a+b-c)=0.$

[The result can of course be written at once as a determinant—see Art. 433.]

Since the given equations are unchanged by interchanging a and x, b and y, c and z, the result of eliminating a, b, c can be written down.

22. We have $$\frac{x-a}{y-b}-\frac{x-b}{y-a}+\frac{y-b}{x-a}-\frac{y-a}{x-b}=0;$$

$$\therefore \frac{(a-b)(a+b-x-y)}{(y-a)(y-b)}+\frac{(a-b)(x+y-a-b)}{(x-a)(x-b)}=0.$$

Hence, either $a=b$ or $x+y=a+b$, or

$$\frac{1}{(y-a)(y-b)}-\frac{1}{(x-a)(x-b)}=0,$$

giving $(x-y)(x+y-a-b)=0.$

23. $$\Sigma\frac{a^2}{b-c}=\frac{\Sigma a^2(c-a)(a-b)}{(b-c)(c-a)(a-b)}=\frac{-\Sigma a^4+\Sigma a^3b-abc\Sigma a}{(b-c)(c-a)(a-b)}.$$

$$\Sigma\frac{b-c}{a^2}=\frac{\Sigma b^2c^2(b-c)}{a^2b^2c^2}=-\frac{1}{a^2b^2c^2}(b-c)(c-a)(a-b)\Sigma bc.$$

Hence $$\Sigma\frac{a^2}{b-c}\,.\,\Sigma\frac{b-c}{a^2}=\frac{1}{a^2b^2c^2}\Sigma bc\,(\Sigma a^4-\Sigma a^3b).$$

Now $$\begin{aligned}\Sigma a^4-\Sigma a^3b&=\Sigma a\,.\,\Sigma a^3-\Sigma a^3b-\Sigma a^3b\\&=-2\Sigma a^3b=-2\Sigma ab\,.\,\Sigma a^2+2\Sigma abc^2\\&=-2\Sigma ab\,.\,\{(\Sigma a)^2-2\Sigma ab\}+2abc\Sigma a\\&=4(\Sigma ab)^2.\end{aligned}$$

Thus $$\Sigma\frac{a^2}{b-c}\,.\,\Sigma\frac{b-c}{a^2}=\frac{4}{a^2b^2c^2}(\Sigma bc)^3=4abc\left(\Sigma\frac{1}{a}\right)^3.$$

24. The necessary and sufficient condition is that

$$(\lambda b+\mu b')^2-4(\lambda a+\mu a')(\lambda c+\mu c')>0,$$

or $$\lambda^2(b^2-4ac)+2\lambda\mu\{bb'-2(ac'+a'c)\}+\mu^2(b'^2-4a'c')>0.$$

Hence [Art. 131], the conditions required are $b^2-4ac>0$,

and $$(b^2-4ac)(b'^2-4a'c')-\{bb'-2(ac'+a'c)\}^2>0,$$

i. e. $4bb'(ac'+a'c)-4b^2a'c'-4b'^2ac-4(ac'-a'c)^2>0$

i. e. $(bc'-b'c)(ab'-a'b)-(ac'-a'c)^2>0.$

25. The equations are

$$x(b+c-2a)-y(c+a-2b)=2(b-a),$$
$$x(e+f-2d)-y(f+d-2e)=2(e-d).$$

These are indeterminate if

$$\frac{b+c-2a}{e+f-2d}=\frac{c+a-2b}{f+d-2e}=\frac{b-a}{c-d}.$$

Either of the first two fractions is equal to the third by Art. 113, and also equal to

$$\frac{2(b+c-2a)+(c+a-2b)}{2(e+f-2d)+(f+d-2e)}=\frac{c-a}{f-d}.$$

Thus we have the single condition

$$(b-a)/(e-d)=(c-a)/(f-d).$$

26. From (i) $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2=0.$

Hence, from (ii), putting A, B, C for $b-c$, $c-a$, $a-b$ respectively, so that $A+B+C=0$;

$$\Sigma Ax\left(\frac{y^3}{B}+\frac{z^3}{C}\right)+2\Sigma y^2z^2=0;$$

$$\therefore\ \Sigma\frac{x^2}{A}(Bxy+Cxz)+2\Sigma y^2z^2=0.$$

But $$2xy=z^2-x^2-y^2,\ \&c.$$

Hence $$\Sigma\frac{x^2}{A}\{B(z^2-x^2-y^2)+C(y^2-z^2-x^2)\}+4\Sigma y^2z^2=0;$$

$$\therefore\ \Sigma\frac{x^2}{A}\{(B-C)z^2-(B-C)y^2+Ax^2\}+4\Sigma y^2z^2=0;$$

$$\therefore\ \Sigma x^4+\Sigma y^2z^2\left(\frac{A-C}{B}+\frac{A-B}{C}\right)+4\Sigma y^2z^2=0;$$

$$\therefore\ \Sigma y^2z^2\left(\frac{A-C}{B}+\frac{A-B}{C}+6\right)=0,$$

$$\text{i.e. } \Sigma-\frac{2}{BC}(B-C)^2y^2z^2=0;$$

$$\therefore\ \Sigma A(B-C)^2/x^2=0.$$

27. From $$x+y+z=0=a^2x+b^2y+c^2z,$$

we have $$\frac{x}{b^2-c^2}=\frac{y}{c^2-a^2}=\frac{z}{a^2-b^2}.$$

Hence each fraction is equal to

$$\frac{\Sigma a^3x}{\Sigma a^3(b^2-c^2)}=\frac{\Sigma bcx}{\Sigma bc(b^2-c^2)},$$

$$\text{i.e. } \frac{\Sigma a^3x}{-(b-c)(c-a)(a-b)\Sigma bc}=\frac{\Sigma bcx}{-(b-c)(c-a)(a-b)\Sigma a};$$

$$\therefore\ \Sigma a\,.\,\Sigma a^3x=\Sigma bc\,.\,\Sigma bcx.$$

28. (i) $\sqrt{2x^2-1}-\sqrt{x^2-x+2}=\sqrt{2x^2+2x+3}-\sqrt{x^2-3x-2}$.

Squaring, we have

$$\sqrt{2x^2-1}\,\sqrt{x^2-x+2}=\sqrt{2x^2+2x+3}\,\sqrt{x^2-3x-2};$$
$$\therefore\ 2x^4-2x^3+3x^2+x-2=2x^4-4x^3-7x^2-13x-6;$$
$$\therefore\ x^3+5x^2+7x+2=0,$$
$$\text{i.e. } (x+2)(x^2+3x+1)=0.$$

(ii) $$\frac{(a-x)^4+(b-x)^4}{(a-x)^2+(b-x)^2}=\{(a-x)-(b-x)\}^2;$$

$$\therefore\ (a-x)^4+(b-x)^4=\{(a-x)^2+(b-x)^2\}\{(a-x)^2+(b-x)^2-2(a-x)(b-x)\};$$
$$\therefore\ 2(a-x)^2(b-x)^2-2(a-x)(b-x)\{(a-x)^2+(b-x)^2\}=0.$$
$$\therefore\ (a-x)(b-x)\{(a-x)^2+(b-x)^2-(a-x)(b-x)\}=0.$$

(iii) $$\left\{\frac{x-a}{x+a}+\frac{(x-b)(x-c)}{(x+b)(x+c)}\right\}^2=1-\left(\frac{x-b}{x+b}\right)^2-\left(\frac{x-c}{x+c}\right)^2+\frac{(x-b)^2(x-c)^2}{(x+b)^2(x+c)^2}$$
$$=\left\{1-\frac{(x-b)^2}{(x+b)^2}\right\}\left\{1-\frac{(x-c)^2}{(x+c)^2}\right\}.$$

Whence $$\{2x^3-2x(ca+ab-bc)\}^2=16bcx^2(x+a)^2.$$

Hence $x^2=0$, or

$$x^2-ca-ab+bc=\pm 2\sqrt{bc}\,.\,(x+a);$$
$$\therefore\ x=\pm\sqrt{bc}\pm\sqrt{ca}\pm\sqrt{ab},$$

all the signs of the ambiguities being positive or only one positive.

(iv) From the first two equations

$$\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}=\lambda, \text{ say.}$$

Hence $$\lambda^3\{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3\}=3(b-c)(c-a)(a-b),$$
whence $$abc\lambda^3=1.$$

(v) We have
$$(2y-1)(2z-1)=1,$$
$$(2z-1)(2x-1)=1,$$
$$(2x-1)(2y-1)=1.$$

Hence $$2x-1=\pm 1, \text{ \&c.}$$

Thus the values are 0, 0, 0; or 1, 1, 1.

(vi) We have
$$(x+y)(x+z)=a^2,$$
$$(y+z)(y+x)=b^2,$$
$$(z+x)(z+y)=c^2.$$

Hence $$(y+z)^2=b^2c^2/a^2,$$
$$\therefore\ y+z=\pm bc/a, \text{ \&c.}$$
$$\therefore\ 2x=\pm\left\{\frac{ca}{b}+\frac{ab}{c}-\frac{bc}{a}\right\}, \text{ \&c.}$$

(vii) From $x^2(x^2+4)=y^2(y^2+4)$,

$$(x^2-y^2)(x^2+y^2+4)=0.$$

Also $(y^2-z^2)(y^2+z^2+4)=0.$

Hence $x^2=y^2=z^2$, and therefore $x^2(x^2+4)=\pm 5x^3$.

Hence $x=0$ or $x^2\mp 5x+4=0$,

giving $x=\pm 4$ or $x=\pm 1$.

Thus the real values are 0, 0, 0; ± 4, ± 4, ± 4; ± 1, ± 1, ± 1.

[The ambiguities being so taken that xyz is positive.]

No real or *imaginary* solutions can be obtained from $x^2+y^2+4=0$.

(viii) Put $x=1+\alpha$, $y=1+\beta$ and $z=1+\gamma$; then we have

$$a(1+\beta)(1+\gamma)+b(1+\beta)+c(1+\gamma)=a+b+c,$$

i.e. $$a\beta\gamma+(a+b)\beta+(a+c)\gamma=0.$$

Also $$b\gamma\alpha+(b+c)\gamma+(b+a)\alpha=0,$$

and $$c\alpha\beta+(c+a)\alpha+(c+b)\beta=0.$$

Hence one solution is $\alpha=\beta=\gamma=0$. Moreover if one of the three quantities α, β, γ is zero they must all three be zero.

If $\alpha\beta\gamma$ be not zero, the equations are equivalent to

$$a+\frac{a+b}{\gamma}+\frac{a+c}{\beta}=0,$$

$$b+\frac{b+c}{\alpha}+\frac{b+a}{\gamma}=0,$$

$$c+\frac{c+a}{\beta}+\frac{c+b}{\alpha}=0.$$

Whence $$b+c-a+2\frac{b+c}{a}=0,$$

giving $$a=2(b+c)/(a-b-c), \text{ \&c.}$$

Hence $$x=1+\alpha=1+\frac{2(b+c)}{a-b-c}=\frac{a+b+c}{a-b-c}.$$

Thus $$x=y=z=1,$$

or $$(a-b-c)x=(b-c-a)y=(c-a-b)z=a+b+c.$$

(ix) $$x-y=2,$$

$$xz-yw=3,$$

$$xz^2-yw^2=5,$$

$$xz^3-yw^3=9.$$

$$(x-y)(xz^2-yw^2)-(xz-yw)^2=10-9,$$

i.e. $$-xy(z-w)^2=1.$$

So $$-xyzw(z-w)^2=9\times 3-5^2=2,$$

and $$-xy(z+w)(z-w)^2=9\times 2-5\times 3=3.$$

Hence $wz=2$ and $w+z=3$,

whence $w=1,\ z=2$ or $w=2,\ z=1$.

Hence $xy=-1$ and $x-y=2$,

whence $x=1,\ y=-1$.

Thus $x=1,\ y=-1,\ z=1,\ w=2$,

or $x=1,\ y=-1,\ z=2,\ w=1$.

(x) From the first two equations we have

$$(x+y)\left\{\frac{a}{c-a}y+\frac{b}{a-b}x\right\}-\frac{c}{b-c}xy=0;$$

$$\therefore\ b(c-a)(b-c)x^2+a(a-b)(b-c)y^2$$
$$-xy\{c(c-a)(a-b)-b(b-c)(c-a)-a(b-c)(a-b)\}=0,$$

i.e. $\{b(c-a)x-a(b-c)y\}\{(b-c)x-(a-b)y\}=0$.

Hence I. $b(c-a)x-a(b-c)y=0$, whence

$$\frac{x}{a(b-c)}=\frac{y}{b(c-a)}=\frac{x+y}{c(b-a)}=\frac{z}{c(a-b)}=\lambda \text{ suppose.}$$

Then, from the third equation we find that $\lambda=1$.

Or II. $(b-c)x-(a-b)y=0$, whence

$$\frac{x}{a-b}=\frac{y}{b-c}=\frac{x+y}{a-c}=\frac{z}{c-a}=\mu,$$

where μ is given by

$$\mu\left\{\frac{b-c}{a-b}+\frac{c-a}{b-c}+\frac{a-b}{c-a}\right\}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

(xi) We have

$$(x+y)(az+cx)-(z+x)(bx+ay)=0,$$

i.e. $x\{-(b-c)x+(c-a)y+(a-b)z\}=0$.

Similarly $y\{(b-c)x-(c-a)y+(a-b)z\}=0$.

Whence it follows that either $x=0$, $y=0$ or $z=0$, or else $x=y=z=0$.

If $x=0$ and $yz\neq 0$, we have

$$a=\frac{cy+bz}{y+z}=y+z,$$

whence $y=a\dfrac{b-a}{b-c}$ and $z=a\dfrac{c-a}{c-b}$.

Thus we have the solutions

$$0,\ 0,\ 0;\ 0,\ a\frac{b-a}{b-c},\ a\frac{c-a}{c-b};\ b\frac{a-b}{a-c},\ 0,\ b\frac{c-b}{c-a};$$

$$c\frac{a-c}{a-b},\ c\frac{b-c}{b-a},\ 0.$$

(xii) Multiply by x, y, z in order; then

$$xyz + bcx^2 + abxy + acxz = 0,$$
$$xyz + bayx + cay^2 + bcyz = 0,$$
$$xyz + cazx + cbzy + abz^2 = 0.$$

Hence $$bc\,(x^2 - yz) = ca\,(y^2 - zx) = ab\,(z^2 - xy).$$

Hence $$x^2 - yz = a\lambda,$$
$$y^2 - zx = b\lambda,$$
$$z^2 - xy = c\lambda;$$

$$\therefore\ (x^2 - yz)^2 - (y^2 - zx)\,(z^2 - xy) = (a^2 - bc)\,\lambda^2,$$

i.e. $$x\,(x^3 + y^3 + z^3 - 3xyz) = (a^2 - bc)\,\lambda^2.$$

Thus $$\frac{x}{a^2 - bc} = \frac{y}{b^2 - ca} = \frac{z}{c^2 - ab} = \mu,$$

where μ is to be obtained by substituting in either of the original equations; and it will be found that $\mu = 0$, or

$$\mu\,(b^2 - ca)\,(c^2 - ab) + bc\,(a^2 - bc) + ab\,(b^2 - ca) + ac\,(c^2 - ab) = 0,$$

giving $\mu = 1$.

Thus $$x = y = z = 0 \text{ or } \frac{x}{a^2 - bc} = \frac{y}{b^2 - ca} = \frac{z}{a^2 - bc} = 1.$$

29. Since α, β, γ are the three values of λ given by the cubic

$$\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1,$$

i.e. $$\lambda^3 + \lambda^2\,(a + b + c - x - y - z)$$
$$+ \lambda\,\{\Sigma bc - \Sigma\,(b + c)\,x\} + abc - bcx - cay - abz = 0,$$

it follows that

$$\alpha\beta\gamma = -abc + bc\,.\,x + cay + abz,$$

or $$x/a + y/b + z/c = 1 + \alpha\beta\gamma/abc.$$

30. We have, if the equations be simultaneously true,

$$\frac{a}{a'}\,x + \frac{a'}{x} = \frac{b}{b'}\,y - \frac{b'}{y},$$
$$\frac{b}{b'}\,y + \frac{b'}{y} = \frac{c}{c'}\,z - \frac{c'}{z},$$
$$\frac{c}{c'}\,z + \frac{c'}{z} = \frac{a}{a'}\,x - \frac{a'}{x}.$$

Hence, by squaring and adding, we have

$$2a + 2b + 2c = -2b - 2c - 2a;$$
$$\therefore\ a + b + c = 0.$$

31. $$p + \omega q + \omega^2 r = (a + \omega c + \omega^2 b)\,(x + \omega^2 y + \omega z),$$

and $$p + \omega^2 q + \omega r = (a + \omega^2 c + \omega b)\,(x + \omega y + \omega^2 z).$$

Hence $$p^2 + q^2 + r^2 - qr - rp - pq$$
$$= (a^2 + b^2 + c^2 - bc - ca - ab)\,(x^2 + y^2 + z^2 - yz - zx - xy) = 0.$$

32. Let x, y be the numbers; then

$$xy = 12(x+y); \quad \therefore \; x = \frac{12y}{y-12} = 12 + \frac{144}{y-12}.$$

Hence y is a positive integer less than 12 and such that $y-12$ is a factor of 144. The possible values of y are 156, 84, 60, 48, 36, 30, 28, 24, 21, 20, 18, 16, 15, 14, 13, the corresponding values of x being 13, 14, 15, 16, 18, &c.

33. We have $n^3x^3 + n^3y^3 - (lx+my)^3 - an^2xy\,(lx+my) = 0$;

$$\therefore \; \frac{x_1x_2x_3}{y_1y_2y_3} = \frac{m^3-n^3}{n^3-l^3}.$$

Hence $$\frac{x_1x_2x_3}{m^3-n^3} = \frac{y_1y_2y_3}{n^3-l^3} = (\text{similarly}) \; \frac{z_1z_2z_3}{l^3-m^3},$$

whence it follows that $x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0$.

34. (i) $a+b+c = \Sigma x^2 + \Sigma yz = (\Sigma x)^2 - 2\Sigma yz + \Sigma yz = {p_1}^2 - p_2$.

(ii) $$\begin{aligned} bc+ca+ab &= \Sigma\,(y^2+zx)\,(z^2+xy) \\ &= \Sigma y^2z^2 + \Sigma xy^3 + xyz\Sigma x \\ &= \{(\Sigma yz)^2 - 2xyz\Sigma x\} + \{\Sigma xy \,.\, \Sigma y^2 - \Sigma xyz^2\} + xyz\Sigma x \\ &= {p_2}^2 - 2p_1p_3 + p_2\,({p_1}^2 - 2p_2) - p_1p_3 + p_1p_3 \\ &= {p_1}^2p_2 - {p_2}^2 - 2p_1p_3. \end{aligned}$$

(iii) $$\begin{aligned} abc &= (x^2+yz)\,(y^2+zx)\,(z^2+xy) \\ &= 2x^2y^2z^2 + \Sigma y^3z^3 + xyz\Sigma x^3. \end{aligned}$$

Now $$\begin{aligned} \Sigma y^3z^3 &= \Sigma yz \,.\, \Sigma y^2z^2 - \Sigma xy^3z^2 \\ &= \Sigma yz\,\{(\Sigma yz)^2 - 2xyz\Sigma x\} - xyz\,\{\Sigma yz \,.\, \Sigma x - 3xyz\} \\ &= p_2\,({p_2}^2 - 2p_1p_3) - p_3\,(p_1p_2 - 3p_3), \end{aligned}$$

and $$\begin{aligned} \Sigma x^3 &= \Sigma x \,.\, \Sigma x^2 - \Sigma xy^2 \\ &= \Sigma x\,\{(\Sigma x)^2 - 2\Sigma xy\} - \{\Sigma xy \,.\, \Sigma x - 3xyz\} \\ &= p_1\,({p_1}^2 - 2p_2) - (p_1p_2 - 3p_3). \end{aligned}$$

35. $x_1 = \dfrac{x_0}{1-x_0}$; $\quad \therefore \; \dfrac{1}{x_1} = \dfrac{1}{x_0} - 1$. So $\dfrac{1}{x_2} = \dfrac{1}{x_1} - 1$, ...,

$$\frac{1}{x_n} = \frac{1}{x_{n-1}} - 1. \quad \text{Hence } \frac{1}{x_n} = \frac{1}{x_0} - n;$$

$$\therefore \quad nx_n = \frac{nx_0}{1-nx_0} = nx_0 + n^2{x_0}^2 + \ldots\ldots$$

36. Let the common difference of the A.P. $x-1$, $y-1$, $z+1$, $u+1$ be d; then $y = x+d$, $z = x-2+2d$, and $u = x-2+3d$.

Then $$x^2 + (x-2+2d)^2 = 2\,(x+d)^2,$$

and $$x^2 + (x-2+3d)^2 = 2\,(x-2+2d)^2.$$

Hence $2x - d^2 + 4d - 2 = 0,$

and $2x(d-2) - d^2 - 4d + 4 = 0.$

Whence $d = 0$ or $d = 1$ or $d = 6,$

the corresponding values of x being $1, -\frac{1}{2}, 7.$

37.

$$\begin{array}{r} 2\,)\,999 \\ 3\,)\,\overline{499-1} \\ 4\,)\,\overline{166-1} \\ 5\,)\,\overline{41-2} \\ 6\,)\,\overline{8-1} \\ \overline{1-2} \end{array}$$

Hence the expression required $= 1 + \underline{|2} + 2\,\underline{|3} + \underline{|4} + 2\,\underline{|5} + \underline{|6}.$

38. $\frac{1}{2}(25+N)$, $\sqrt{25N}$ and $\frac{50N}{25+N}$ must all be positive integers.

The first shews that N must be an *odd* integer, and then the second shews that N must be of the form $(2n+1)^2$.

Then $\frac{50(2n+1)^2}{25+(2n+1)^2}\left[= 50 - \frac{1250}{25+(2n+1)^2}\right]$ must be an integer.

Hence $25+(2n+1)^2$ must be a factor of 1250, and the only factors of 1250 which are greater than 25 are 50, 125, 250, 625 and 1250. Equating these in succession to $25+(2n+1)^2$ we find that 250 and 1250 will alone satisfy the conditions. Thus $N = (2n+1)^2 = 225$ or 1225.

39. The multiplication is as under.

$$\begin{array}{cccc} a & b & & c \\ & & & r-1 \\ \hline c & b+c-a & & a \end{array}$$

Hence $c(r-1) = a + mr,$

$b(r-1) + m = b + c - a + nr,$

$a(r-1) + n = c.$

The first equation gives $a + c = (c-m)r$, whence it follows that $a + c = r$.
Then the third equation gives $ar + n = a + c = r$, whence $a = 1$ and $n = 0$.
We then have $c = r-1$, $m = r-2$, and $b = 0$.
Thus the digits in order are $1, 0, r-1$.

40. Suppose that a and b are excluded from having given places respectively. Then the place from which a is excluded can be filled by b, and there will then be no restriction on the ways of filling the other $r-1$ places. Hence the conditions can be so satisfied in ${}_{n-1}P_{r-1}$ ways. If however the place from which a is excluded be not filled by b, there are $n-2$ ways of filling that place, and then there will be $n-2$ ways of filling up the place from which b is to be excluded, and there will then be no restriction of the ways of filling up the remaining $r-2$ places. Hence the conditions can be so satisfied in $(n-2)(n-2)\,{}_{n-2}P_{r-2}$ ways.

Thus the number required

$$= {}_{n-1}P_{r-1} + (n-2)\,(n-2)\,{}_{n-2}P_{r-2}$$
$$= \{n-1+(n-2)\,(n-2)\}\,{}_{n-2}P_{r-2}$$
$$= (n^2-3n+3)\,{}_{n-2}P_{r-2}.$$

41. $(1-x)^{2n-1} = 1-(2n-1)\,x+\dfrac{(2n-1)\,(2n-2)}{1\,.\,2}\,x^2-\ldots\ldots$

$(1-x)^{-1} = 1+x+x^2+\ldots\ldots$

Hence $1-(2n-1)+\dfrac{(2n-1)\,(2n-2)}{1\,.\,2}$ – &c. to n terms = coefficient of

x^{n-1} in $(1-x)^{2n-2}$.

42. Let S be the sum; then

$$\underline{|n}\,S+2 = 1+\frac{\underline{|n}}{\underline{|n-1}\,\underline{|1}}+\frac{\underline{|n}}{\underline{|n-2}\,\underline{|2}}+\ldots\ldots+1 = (1+1)^n.$$

Hence $S = (2^n-2)/\underline{|n}$.

43. $\{1-x\,(2-x)\}^{-\frac{1}{2}} = 1+\dfrac{1}{2}\,x\,(2-x)+\dfrac{1\,.\,3}{2\,.\,4}\,x^2\,(2-x)^2$

$$+\ldots+\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n}\,x^n\,(2-x)^n+\ldots$$

The coefficient of $x^n = \dfrac{\underline{|2n}}{\underline{|n}\,\underline{|n}}\,\dfrac{1}{2^{2n}}\cdot 2^n - \dfrac{\underline{|2n-2}}{\underline{|n-1}\,\underline{|n-1}}\,\dfrac{1}{2^{2(n-1)}}\cdot\dfrac{(n-1)}{1}\,2^{n-2}$

$$+\frac{\underline{|2n-4}}{\underline{|n-2}\,\underline{|n-2}}\,\frac{1}{2^{2(n-2)}}\cdot\frac{(n-2)\,(n-3)}{1\,.\,2}\,2^{n-4}-\ldots$$

$$=\frac{1}{2^n}\left\{\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}-\frac{\underline{|2n-2}}{\underline{|n-1}\,\underline{|n-2}\,\underline{|1}}+\frac{\underline{|2n-4}}{\underline{|n-2}\,\underline{|n-4}\,\underline{|2}}-\ldots\right\}.$$

Also the coefficient of x^n in $(1-x)^{-1}$ is **1**.

44. If a be the r^{th} coefficient, we have

$$b = a\frac{n-r+1}{r},$$

$$c = a\frac{(n-r+1)\,(n-r)}{r\,(r+1)},$$

and

$$d = a\frac{(n-r+1)\,(n-r)\,(n-r-1)}{r\,(r+1)\,(r+2)}.$$

Hence $$a+b=a\,(n+1)/r,$$

$$c^2-bd=c^2\left\{1-\frac{r+1}{n-r}\cdot\frac{n-r-1}{r+2}\right\}$$

$$=c^2\,(n+1)/\{(r+2)\,(n-r)\};$$

$$\therefore\ (a+b)\,(c^2-bd)=ac^2\,(n+1)^2/\{r\,(r+2)\,(n-r)\}.$$

Again, $$c+d=c\,(n+1)/(r+2),$$

$$b^2-ac=ac\left\{\frac{n-r+1}{r}\cdot\frac{r+1}{n-r}-1\right\}$$

$$=ac\,(n+1)/\{r\,(n-r)\}.$$

Hence $$(c+d)\,(b^2-ac)=ac^2\,(n+1)^2/\{r\,(r+2)\,(n-r)\}.$$

45. $$(1+x)^n\,(1-x)^{-n}=\{2-(1-x)\}^n\,(1-x)^{-n}$$

$$=2^n\,(1-x)^{-n}-n\,.\,2^{n-1}\,(1-x)^{-n+1}+\frac{n\,(n-1)}{1\,.\,2}\,2^{n-2}\,(1-x)^{-n+2}$$

$$-\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,2^{n-3}\,(1-x)^{-n+3}+\ldots$$

The coefficient of x^r can now be written down in the form required.

46. $$(n^2+1)\,\underline{|n}=n\,\underline{|n+1}-(n-1)\,\underline{|n},$$

$$\ldots\ldots\ldots\ =\ \ldots\ldots\ldots\ldots$$

$$(3^2+1)\,\underline{|3}=3\,\underline{|4}-2\,\underline{|3},$$

$$(2^2+1)\,\underline{|2}=2\,\underline{|3}-1\,\underline{|2},$$

$$(1^2+1)\,\underline{|1}=1\,\underline{|2}.$$

Hence $$S_n=n\,\underline{|n+1}.$$

47. $$(1-x)^{n-1}=1-(n-1)\,x+\frac{(n-1)\,(n-2)}{1\,.\,2}\,x^2-\ldots$$

$$(1-x)^{-2}=1+2x+3x^2+\ldots+nx^{n-1}+\ldots.$$

Hence the coefficient of x^{n-1} in the product is

$$n-(n-1)\,.\,(n-1)+\frac{(n-1)\,(n-2)}{1\,.\,2}\,(n-2)-\ldots$$

But the coefficient of x^{n-1} in $(1-x)^{n-3}$ is zero if n be a positive integer greater than 2.

48. It is easily seen that the first three terms in the expansion of the second expression are $1+p_1{}^2x+p_2{}^2x^2$. We have therefore only to prove that, if n be greater than 3,

$$p_n{}^2-(a^2+a)\,p_{n-1}{}^2-a^2\,(a+1)\,p_{n-2}{}^2+a^3\times p_{n-3}{}^2=0.$$

Now $$p_n-ap_{n-1}-ap_{n-2}=0,$$

$$p_{n-1}-ap_{n-2}-ap_{n-3}=0.$$

Hence $p_n^2 - a^2 p_{n-1}^2 - a^2 p_{n-2}^2 = 2a^2 p_{n-1} p_{n-2}$,

and $p_{n-1}^2 + a^2 p_{n-2}^2 - a^2 p_{n-3}^2 = 2a p_{n-1} p_{n-2}$.

Multiply the second of the last pair of equations by a and subtract the product from the first and we have the relation required.

49. $$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

$$\frac{x^2}{(1-x^2)^2} = x^2 + 2x^4 + 3x^6 + 4x^8 + \ldots.$$

$$\frac{x^3}{(1-x^3)^2} = x^3 + 2x^6 + 3x^9 + 4x^{12} + \ldots.$$

$$\ldots\ldots = \ldots\ldots\ldots$$

Hence the sum $= (x + x^2 + x^3 + \ldots) + 2x^2(1 + x^2 + x^4 + \ldots)$
$+ 3x^3(1 + x^3 + x^6 + \ldots)$
$+ 4x^4(1 + x^4 + x^8 + \ldots)$

$$= \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \ldots.$$

50. $$\left\{1 + \frac{a}{1-a} + \frac{b}{(1-a)(1-b)} + \ldots\ldots + \frac{k}{(1-a)(1-b)\ldots(1-k)}\right\}^2$$
$$= \left\{\frac{1}{(1-a)(1-b)\ldots(1-k)}\right\}^2.$$

Again $$1 + \frac{a(2-a)}{(1-a)^2} = \frac{1}{(1-a)^2}.$$

$$\therefore 1 + \frac{a(2-a)}{(1-a)^2} + \frac{b(2-b)}{(1-a)(1-b)^2} = \frac{1}{(1-a)^2}\left\{1 + \frac{b(2-b)}{(1-b)^2}\right\} = \frac{1}{(1-a)^2(1-b)^2}.$$

Hence the second expression $= \frac{1}{(1-a)^2(1-b)^2\ldots(1-k)^2}$.

51. $$\left(1 - \frac{3}{4}\right)^{-\frac{1}{3}} = 1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \frac{1.4.7.10}{4.8.12.16} + \ldots$$

$$\left(1 + \frac{3}{4}\right)^{-\frac{1}{3}} = 1 - \frac{1}{4} + \frac{1.4}{4.8} - \frac{1.4.7}{4.8.12} + \frac{1.4.7.10}{4.8.12.16} - \ldots.$$

Hence $$\left(1 - \frac{3}{4}\right)^{-\frac{1}{3}} + \left(1 + \frac{3}{4}\right)^{-\frac{1}{3}} = 2S.$$

52. $$S = 1^2 + \frac{3^2}{\lfloor 1} + \frac{5^2}{\lfloor 3} + \frac{7^2}{\lfloor 5} + \ldots$$

$$= 1 + \frac{(1+2)^2}{\lfloor 1} + \frac{(3+2)^2}{\lfloor 3} + \frac{(5+2)^2}{\lfloor 5} + \ldots$$

$$=1+\frac{1}{\lfloor 1}+\frac{3}{\lfloor 2}+\frac{5}{\lfloor 4}+\frac{7}{\lfloor 6}+\cdots$$
$$+4\left\{\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\cdots\right\}$$
$$+4\left\{\frac{1}{\lfloor 1}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\cdots\right\}$$
$$=1+1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 6}+\cdots$$
$$+4\left\{1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\cdots\right\}$$
$$=1+5e.$$

53. In order to fully determine the series the linear relation can only connect *four* terms. We may therefore assume that the series is the expansion of $\dfrac{a+bx+cx^2}{1+px+qx^2+rx^3}$.

Hence we have identically

$$(a+bx+cx^2)=(1+px+qx^2+rx^3)(2+x+4x^2+\ldots);$$
$$\therefore\ a=2,$$
$$b=2p+1,$$
$$c=2q+p+4,$$
$$0=2r+q+4p+19,$$
$$0=r+4q+19p+70,$$
$$0=4r+19q+70p+229.$$

From the last three equations we have

$$r=-3,\ q=7,\ p=-5.$$

Then $a=2,\ b=-9,\ c=13.$

Thus the series is the expansion of

$$\frac{2-9x+13x^2}{1-5x+7x^2-3x^3}=\frac{2-9x+13x^2}{(1-x)^2(1-3x)}=\frac{4}{1-x}-\frac{3}{(1-x)^2}+\frac{1}{(1-3x)}.$$

Hence the n^{th} term of the series is $(4-3n+3^{n-1})\,x^{n-1}$.

54. The series will be the successive coefficients in the expansion of $(\alpha+\beta x)/(1-x-x^2)$ if α and β be properly chosen; and it is easily seen that $\alpha=1=\beta$.

Now
$$\frac{1+x}{1-x-x^2}=\frac{1+x}{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}=\frac{1}{\sqrt{5}-1-x}-\frac{1}{\sqrt{5}+1+x}.$$

The n^{th} term is therefore

$$\left(\frac{1}{\sqrt{5}-1}\right)^n-\left(\frac{1}{\sqrt{5}+1}\right)^n=\frac{1}{4^n}\{(\sqrt{5}+1)^n-(\sqrt{5}-1)^n\}.$$

Let a, b be the last two terms which have r digits, and let c, d, e, f, g, h be the next consecutive terms.

Then
$$c=a+b>10^r,$$
$$d=a+2b,$$
$$e=2a+3b,$$
$$f=3a+5b,$$
$$g=5a+8b,$$
$$h=8a+13b.$$

Then since b is less than $2a$, g must be greater than $10\frac{1}{2}\times b$, and therefore g *may* be greater than 10^{r+1}. Also, since $b>a$ and $a+b>10^r$, h *must* be greater than 10^{r+1}.

55. $$u_n=\frac{2n+1}{(2n-1)\,2n\,(2n+2)}=\frac{(2n+1)^2}{(2n-1)\,2n\,(2n+1)\,(2n+2)}$$
$$=\frac{1}{2n\,(2n+1)}+\frac{1}{2n\,(2n+1)\,(2n+2)}+\frac{4}{(2n-1)\,2n\,(2n+1)\,(2n+2)}.$$

Hence $$S_n=C-\frac{1}{(2n+1)}-\frac{1}{2\,(2n+1)\,(2n+2)}+\frac{4}{6n\,(2n+1)\,(2n+2)}.$$

And
$$\frac{3}{8}=C-\frac{1}{3}-\frac{1}{2\,.\,3\,.\,4}+\frac{4}{6\,.\,3\,.\,4};$$
$$\therefore\ C=\frac{25}{36},\ \text{and}\ S_\infty=C.$$

56. We know [Art. 297, Ex. 3], that
$$\frac{\lfloor n}{(x+1+y)(x+2+y)\ldots(x+n+1+y)}=\frac{1}{x+1+y}-\frac{n}{x+2+y}+\ldots$$
$$+\frac{(-1)^{r-1}C_{r-1}}{x+r+y}+\ldots$$

Hence
$$\frac{\lfloor n}{(x+1)\,(x+2)\ldots(x+n+1)}\cdot\frac{1}{1+y\left(\frac{1}{x+1}+\frac{1}{x+2}+\ldots\right)+\text{higher powers of }y}$$
$$=\frac{1}{x+1}\cdot\frac{1}{1+\frac{y}{x+1}}+\ldots+\frac{(-1)^{r-1}C_{r-1}}{x+r}\,\frac{1}{1+\frac{y}{x+r}}+\ldots$$

Expand in powers of y, and equate the coefficients of y on the two sides of the equation last written; then
$$\frac{\lfloor n}{(x+1)\,(x+2)\ldots(x+1+n)}\left\{\frac{1}{x+1}+\frac{1}{x+2}+\ldots\right\}=\frac{1}{(x+1)^2}+\ldots+\frac{(-1)^{r-1}C_{r-1}}{(x+r)^2}+\ldots$$
This proves the theorem, since we know that
$$\frac{\lfloor n}{(x+1)\,(x+2)\ldots(x+n)}=\frac{1}{x+1}-\frac{n}{x+2}+\ldots+\frac{(-1)^{r-1}C_{r-1}}{x+r}+\ldots$$

57. The coefficient of x^{n+1} in the expansion of $(1-4x)^{\frac{1}{2}}$ is

$$-2\frac{\lfloor 2n}{\lfloor n \lfloor n+1}.$$

Hence $$2y = 1-(1-4x)^{\frac{1}{2}},$$

whence $(2y-1)^2 = 1-4x$, and therefore $y^2-y+x=0$.

Hence also $$8y^3 = 1-3(1-4x)^{\frac{1}{2}}+3(1-4x)-(1-4x)^{\frac{3}{2}}.$$

The coefficient x^{n+2} in the expansion of the expression on the right is

$$3\,.\,2\frac{\lfloor 2n+2}{\lfloor n+1 \lfloor n+2}-3\,.\,4\frac{\lfloor 2n}{\lfloor n \lfloor n+2}=\frac{24\lfloor 2n}{\lfloor n-1 \lfloor n+2}.$$

58. The sum required is the sum of all the terms of r dimensions in the continued product

$$(1+a+a^2+\ldots+a^{m-1})(1+b+b^2+\ldots+b^{m-1})\ldots,$$

i.e. in $$\frac{1-a^m}{1-a}\cdot\frac{1-b^m}{1-b}\cdot\frac{1-c^m}{1-c}\ldots,$$

i. e. in $$[1-\Sigma a^m+\Sigma a^m b^m-\ldots][1+{}_nH_1+{}_nH_2+\ldots].$$

Hence the sum required

$$={}_nH_r-\Sigma a^m\,{}_nH_{r-m}+\Sigma a^m b^m\,{}_nH_{r-2m}-\ldots.$$

59. If H_n stand for the sum of all the homogeneous products of a, b, c of n dimensions; then by Art. 300, Ex. 3,

$$H_n=\Sigma\frac{a^{n+2}}{(a-b)(a-c)};$$

$$\therefore\ \Sigma_{r=0}^{r=n}H_r=\Sigma\frac{a^2+a^3+\ldots+a^{n+2}}{(a-b)(a-c)}$$

$$=\Sigma\frac{a^2(a^{n+1}-1)}{(a-b)(a-c)(a-1)}$$

$$=\Sigma\frac{a^{n+3}}{(a-b)(a-c)(a-1)}-\Sigma\frac{a^2}{(a-b)(a-c)(a-1)}.$$

Now $$\Sigma\frac{a^2}{(a-b)(a-c)(a-1)}=\frac{1}{\Pi(b-c)\,.\,\Pi(a-1)}\Sigma a^2(c-b)(b-1)(c-1)$$

$$=\frac{1}{\Pi(b-c)\,.\,\Pi(a-1)}\{abc\Sigma a(c-b)+\Sigma a^2(c^2-b^2)+\Sigma a^2(c-b)\}$$

$$=\frac{1}{(a-1)(b-1)(c-1)},$$

since $$\Sigma a(c-b)=0=\Sigma a^2(c^2-b^2),\text{ and }\Sigma a^2(c-b)=\Pi(b-c).$$

Hence $$\Sigma_{r=}^{r=n}H_r=\Sigma\frac{a^{n+3}}{(a-b)(a-c)(a-1)}-\frac{1}{(a-1)(b-1)(c-1)}.$$

60. The envelopes being taken up in any order, let the corresponding bills and cheque be numbered

$$b_1,\ b_2,\ b_3, \ldots,\ b_n,$$

and

$$c_1,\ c_2,\ c_3, \ldots,\ c_n.$$

Then the ways of filling up the first envelope in the required manner is represented by the terms of

$$(b_1+b_2+b_3+\ldots)\,(c_1+c_2+c_3+\ldots)-b_1c_1.$$

Similarly for all the other envelopes. Hence as each bill and each cheque can only be taken once, the number of ways required is the coefficient of $b_1b_2b_3\ldots b_nc_1c_2c_3\ldots c_n$ in the continued product

$$(BC-b_1c_1)\,(BC-b_2c_2)\ldots(BC-b_nc_n),$$

where

$$B=b_1+b_2+\ldots+b_n,$$

and

$$C=c_1+c_2+\ldots+c_n.$$

Thus we have to find the coefficient of $b_1b_2\ldots b_n\,.\,c_1c_2\ldots c_n$ in

$$B^nC^n-\Sigma b_1c_1\,.\,B^{n-1}C^{n-1}+\ldots+(-1)^r\,\Sigma b_1b_2\ldots b_rc_1c_2\ldots c_r\,.\,B^{n-r}C^{n-r}+\ldots$$

Now there are $\dfrac{\lfloor n}{\lfloor r\,\lfloor n-r}$ terms in $\Sigma b_1b_2\ldots b_rc_1c_2\ldots c_r$, and corresponding to each of these terms there will be $\lfloor n-r\,.\,\lfloor n-r$ ways of taking the remaining letters in $B^{n-r}C^{n-r}$. Thus the coefficient of $b_1b_2\ldots b_nc_1c_2\ldots c_n$ in

$$(-1)^r\,\Sigma b_1b_2\ldots b_rc_1c_2\ldots c_rB^{n-r}C^{n-r}$$

will be

$$(-1)^r\frac{\lfloor n}{\lfloor r\,\lfloor n-r}\times\lfloor n-r\,.\,\lfloor n-r.$$

Hence the result required is

$$\lfloor n\,\lfloor n-\frac{\lfloor n\,\lfloor n-1}{\lfloor 1}+\frac{\lfloor n\,\lfloor n-2}{\lfloor 2}-\ldots+(-1)^r\,.\,\frac{\lfloor n\,\lfloor n-r}{\lfloor r}+\ldots$$

[S. John's College, 1889.]

61. The n^{th} circle is cut by each of the remaining circles in two points, so that the circumference is divided into $2(n-1)$ parts. Hence if the n^{th} circle were absent, there would be $2(n-1)$ fewer divisions of the plane. Thus, if $f(n)$ denote the number of parts into which the plane is divided by n circles, we have

$$f(n)=2(n-1)+f(n-1).$$

So also

$$f(n-1)=2(n-2)+f(n-2)$$

$$\ldots\ldots\ldots=\ldots\ldots\ldots\ldots$$

$$f(3)=2\,.\,2+f(2),$$

$$f(2)=2\,.\,1+f(1),$$

and

$$f(1)=2.$$

Hence

$$f(n)=2+2\{1+2+\ldots(n-1)\}$$

$$=2+(n-1)\,n.$$

62. Let $\dfrac{x^{n-1}}{(x+1)(x+2)\ldots(x+n)} = \dfrac{a_1}{x+1} + \dfrac{a_2}{x+2} + \ldots + \dfrac{a_r}{x+r} + \ldots + \dfrac{a_n}{x+n}$.

Then $x^{n-1} = \Sigma a_r (x+1)(x+2)\ldots(x+r-1)(x+r+1)\ldots(x+n)$.

In this identity put $x+r=0$; then

$$(-r)^{n-1} = a_r(-r+1)(-r+2)\ldots(-1).1.2\ldots(n-r).$$

Hence $a_r = (-1)^{n-r} r^{n-1} / \{\underline{|r-1} \cdot \underline{|n-r}\}$.

Thus $a_n = n^{n-1}/(n-1)!$

$a_{n-1} = -(n-1)^{n-1}/\{(n-2)!\,1!\}$,

$a_{n-2} = (n-2)^{n-1}/(n-3)!\,2!\}$, &c.

Hence

$$\frac{x^{n-1}\,|\,n}{(x+1)(x+2)\ldots(x+n)} = \frac{n^n}{x+n} - \frac{n}{1}\cdot\frac{(n-1)^n}{x+n-1} + \frac{n(n-1)}{1.2}\cdot\frac{(n-2)^n}{x+n-2} - \ldots\ldots.$$

63. (i) $$u_n = \frac{n}{n^2+1} > \frac{1}{2n}.$$

$$\therefore\ \Sigma u_n > \frac{1}{2}\Sigma\frac{1}{n}.$$

Hence Σu_n is divergent.

(ii) $$u_n = \frac{n}{n^3-1} < \frac{2}{n^2}.$$

But $\Sigma\dfrac{1}{n^2}$ is convergent; $\therefore$ Σu_n is convergent.

(iii) $$u_n/u_{n-1} = \frac{n^2-n+1}{n^4-n^2+1}\cdot\frac{(n-1)^4-(n-1)^2+1}{(n-1)^2-(n-1)+1}\cdot a.$$

Hence $$\text{Lt}\, u_n/u_{n-1} = a.$$

Hence, if $a>1$, u_n/u_{n-1} will, either from the beginning or after a *finite* number of terms, be greater than 1 for all values of n. And if $a<1$, u_n/u_{n-1} will, either from the beginning or after a *finite* number of terms, be less than 1 for all values of n.

Thus Σu_n is divergent or convergent according as a is greater or less than 1.

If $a=1$, the series becomes

$$\Sigma\frac{n^2-n+1}{n^4-n^2+1} < \frac{1}{n^2} \text{ (since } -n^3+2n^2-1<1).$$

Hence $\Sigma\dfrac{n^2-n+1}{n^4-n^2+1}$ is convergent.

(iv) $$u_{n+1}/u_n = \frac{(3n+2)(3n+4)}{3(n+1)\,3(n+1)};$$

$$\therefore\ n\left(1-\frac{u_{n+1}}{u_n}\right) = n\left\{1-\frac{(3n+2)(3n+4)}{9(n+1)^2}\right\}$$

$$= \frac{n}{9(n+1)^2} < 1.$$

Hence Σu_n is divergent. [Art. 342.]

(v) $$u_n = n \log \frac{2n+1}{2n-1} - 1 = n \log \left(1 + \frac{2}{2n-1}\right) - 1$$

$$= n \left\{ \frac{2}{2n-1} - \frac{1}{2} \cdot \left(\frac{2}{2n-1}\right)^2 + \frac{1}{3} \left(\frac{2}{2n-1}\right)^3 + \dots \right\} - 1$$

$$= -\frac{1}{(2n-1)^2} + \frac{n}{3} \left(\frac{2}{2n-1}\right)^3 - \frac{n}{4} \left(\frac{2}{2n-1}\right)^4 + \dots$$

Hence $$-u_n < \frac{1}{(2n-1)^2}, \text{ if } n > 1.$$

Hence $$-\Sigma u_n < \Sigma \frac{1}{(2n-1)^2}.$$

Hence $\sum_{n=2}^{n=\infty} u_n$ is negative and convergent.

64. $$\frac{u_{r+1}}{u_r} = \frac{m+n}{m-n} \cdot \frac{(r+1)m - rn}{(r+1)m + rn}.$$

Hence $$\mathrm{Lt}_{r=\infty}\, r\left(1 - \frac{u_{r+1}}{u_r}\right) = \mathrm{Lt}_{r=\infty} \frac{-2mnr}{(m-n)\{(r+1)m + rn\}}$$

$$= -\frac{2mn}{m^2 - n^2}.$$

The series is therefore divergent or convergent according as $\frac{2mn}{n^2-m^2}$ is less or greater than unity, i.e. according as $m \lesseqgtr (\sqrt{2}-1)\, n$.

65. Let $$u = \left\{ \frac{(m+1)(m+2)\dots(m+n)}{1 \cdot 2 \cdot 3 \dots n} \right\}^{\frac{1}{n}}.$$

Then $$\log u = \frac{1}{n} \left\{ \log\left(1 + \frac{m}{1}\right) + \log\left(1 + \frac{m}{2}\right) + \dots + \log\left(1 + \frac{m}{n}\right) \right\}$$

$$= \frac{1}{n} \left\{ m \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}\right) - \frac{m^2}{2} \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}\right) \right.$$

$$\left. + \frac{m^3}{3} \left(\frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{n^3}\right) - \dots \right\}.$$

Hence, when n is infinite,

$$\log u = \mathrm{Lt}_{n=\infty} \frac{m}{n} \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}\right).$$

Now, from Ex. XXXI. 21, we have

$$\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} > \log(1+n),$$

and

$$\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} < \log(1+n) + 1.$$

Hence
$$\frac{\log u}{m} > \mathrm{Lt}_{n=\infty} \frac{\log(1+n)}{n},$$
$$< \mathrm{Lt}_{n=\infty} \frac{\log(1+n)+1}{n}.$$

Now, if $\log(1+n)/n = x$, we have
$$1+n = e^{nx} = 1+nx+\frac{n^2x^2}{1.2}+\frac{n^3x^3}{1.2.3}+\ldots,$$
whence it follows that $x=0$ when $n=\infty$.

Thus, when $n=\infty$, $\frac{1}{m}\log u = 0$, and therefore $u=1$.

66.
$$2\sqrt{n^2+1} = 2n+2\{\sqrt{n^2+1}-n\}$$
$$= 2n+\frac{2}{\sqrt{n^2+1}+n}.$$

Also
$$\sqrt{n^2+1}+n = 2n+\{\sqrt{n^2+1}-n\}$$
$$= 2n+\frac{1}{\sqrt{n^2+1}+n}.$$

Hence
$$2\sqrt{n^2+1} = 2n+\frac{2}{2n}+\frac{1}{2n}+\frac{1}{2n}+\ldots$$
$$= 2n+\frac{1}{n}+\frac{1}{4n}+\frac{2}{2n}+\ldots$$
$$= 2n+\frac{1}{n}+\frac{1}{4n}+\frac{1}{n}+\frac{1}{4n}+\ldots$$

Or thus. Let
$$x = \frac{1}{n}+\frac{1}{4n}+\frac{1}{n}+\frac{1}{4n}+\ldots$$
$$= \frac{1}{n}+\frac{1}{4n+x} = \frac{4n+x}{4n^2+nx+1};$$
$$\therefore\ nx^2+4n^2x-4n=0,$$
whence
$$x = -2n+2\sqrt{n^2+1};$$
$$\therefore\ 2\sqrt{n^2+1} = 2n+x = 2n+\frac{1}{n}+\frac{1}{4n}+\ldots$$

67. Put
$$\frac{a\phi(x+1)}{x\phi(x)} = \frac{a}{x+F};$$
then
$$F = x\frac{\phi(x)-\phi(x+1)}{\phi(x+1)}.$$

But $$\phi(x)-\phi(x+1)=\frac{a}{x(x+1)}+\frac{a^2}{1!}\cdot\frac{1}{x(x+1)(x+2)}$$
$$+\frac{a^3}{2!}\cdot\frac{1}{x(x+1)(x+2)(x+3)}+\dots$$
$$=\frac{a}{x(x+1)}\phi(x+2);$$
$$\therefore F=\frac{a}{(x+1)}\frac{\phi(x+2)}{\phi(x+1)}.$$

Hence $$\frac{a\phi(x+1)}{x\phi(x)}=\cfrac{a}{x+\cfrac{a}{x+1}\cfrac{\phi(x+2)}{\phi(x+1)}}$$
$$=\frac{a}{x}+\frac{a}{x+1}+\frac{a}{x+2}+\dots$$

68. $$\frac{p_n}{q_n}=\frac{lp_{n-1}+p_{n-2}}{lq_{n-1}+q_{n-2}}.$$
$$F=\frac{\left(l+\frac{p_n}{q_n}\right)p_{n-1}+p_{n-2}}{\left(l+\frac{p_n}{q_n}\right)q_{n-1}+q_{n-2}}$$
$$=\frac{q_n(lp_{n-1}+p_{n-2})+p_np_{n-1}}{q_n(lq_{n-1}+q_{n-2})+p_nq_{n-1}}$$
$$=\frac{q_np_n+p_np_{n-1}}{q_n^2+p_nq_{n-1}}$$

69. We have $$p_n=(n+1)p_{n-1}-np_{n-2};$$
i.e. $$p_n-p_{n-1}=n\{p_{n-1}-p_{n-2}\};$$
$$\therefore p_{n-1}-p_{n-2}=(n-1)\{p_{n-2}-p_{n-3}\}$$
$$\dots\dots\dots\dots=\dots\dots\dots\dots$$
$$p_2-p_1=2.$$

Hence $$p_n-p_{n-1}=\underline{|n}.$$

Hence $$p_n=\underline{|n}+\underline{|n-1}+\underline{|n-2}+\dots+\underline{|2}+\underline{|1}.$$

Similarly $$q_n-q_{n-1}=\underline{|n}.$$

Hence $$q_n=\underline{|n}+\underline{|n-1}+\dots+\underline{|2}+q_1$$
$$=\underline{|n}+\underline{|n-1}+\dots+\underline{|2}+\underline{|1}+1.$$

70. $$p_n=(n^2+1)p_{n-1}-n^2p_{n-2}.$$

Hence $$p_n-p_{n-1}=n^2(p_{n-1}-p_{n-2})$$
$$=n^2(n-2)^2\dots3^2(p_2-p_1)$$
$$=(n!)^2.$$

Hence $p_n=(n!)^2+\ldots+(3!)^2+(2!)^2+p_1$.

Also $q_n=(n!)^2+\ldots+(3!)^2+(2!)^2+q_1$.

Thus $$\frac{p_n}{q_n}=\frac{(n!)^2+\ldots+(2!)^2+(1!)^2}{(n!)^2+\ldots+(2!)^2+(1!)^2+1}.$$

71. This is Art. 370 [E].

Or thus $$p_n=(a_n+1)\,p_{n-1}-a_np_{n-2};$$

$$\therefore\ p_n-p_{n-1}=a_n(p_{n-1}-p_{n-2})$$
$$=a_na_{n-1}(p_{n-2}-p_{n-3})=\ldots$$
$$=a_na_{n-1}\ldots a_3(p_2-p_1)$$
$$=a_na_{n-1}\ldots a_2a_1.$$

Hence $p_n=a_na_{n-1}\ldots a_1+a_{n-1}a_{n-2}\ldots a_1+\ldots+a_2a_1+p_1$.

Similarly $q_n=a_na_{n-1}\ldots a_1+\ldots+a_2a_1+q_1$.

Hence $$p_n/q_n=\sigma_n/(\sigma_n+1).$$

72. In Art. 370 [C], put $a_1=1$, $a_2=a$, $a_3=a^2$, &c.; also put $b_1=1$, $b_2=1-a$, $b_3=1-a^2$, &c. Then we have

$$1+\frac{a}{1-a}+\frac{a}{1-a}\frac{a^2}{1-a^2}+\frac{a}{1-a}\frac{a^2}{1-a^2}\frac{a^3}{1-a^3}+\ldots$$
$$=\frac{1}{1}-\frac{a}{1}-\frac{a^2(1-a)}{1}-\frac{a^3(1-a^2)}{1}-\ldots$$

The identity $(1+a)(1+a^2)(1+a^3)\ldots$ to infinity

$$=1+\frac{a}{1-a}+\frac{a}{1-a}\frac{a^2}{1-a^2}+\ldots \text{ to infinity}$$

follows at once from Ex. xxix. 28.

73. Since $\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_3}-\frac{1}{a_4}+\ldots=\frac{1}{a_1}+\frac{a_1^2}{a_2-a_1}+\frac{a_2^2}{a_3-a_2}+\ldots$.

Put $a_1=(a-1)(2a-1)$, $a_2=(2a-1)(3a-1)$, $a_3=(3a-1)(4a-1)$, &c.

Then $$\left\{\frac{1}{(a-1)(2a-1)}-\frac{1}{(2a-1)(3a-1)}+\ldots \text{ to } n \text{ terms}\right\}$$

will be equal to the n^{th} convergent of

$$\frac{1}{(a-1)(2a-1)}+\frac{(a-1)^2(2a-1)^2}{2a(2a-1)}+\frac{(2a-1)^2(3a-1)^2}{2a(3a-1)}+\ldots,$$

that is, of

$$\frac{1}{a-1}\left\{\frac{1}{2a-1}+\frac{(a-1)(2a-1)}{2a}+\frac{(2a-1)(3a-1)}{2a}+\ldots\right\}.$$

74. Since the first convergents to the square roots of two consecutive odd numbers differ by unity, the even number between must be a perfect square.

Hence the odd numbers are of the form $4n^2-1$ and $4n^2+1$ respectively.

Then $$\sqrt{(4n^2-1)} = 2n-1+\frac{1}{1}+\frac{1}{4n-2}+\frac{1}{1}+\frac{1}{4n-2}+\cdots$$

and $$\sqrt{(4n^2+1)} = 2n+\frac{1}{4n}+\frac{1}{4n}+\cdots.$$

The third convergents are

$$2n-1+\frac{4n-2}{4n-1} \text{ and } 2n+\frac{4n}{16n^2+1}.$$

Hence $$\frac{4n}{16n^2+1}+\frac{1}{4n-1}=\frac{497}{3855};$$

$$\therefore\ 64\,.\,497n^3+\ldots-64\,.\,4\,.\,17=0.$$

Hence [Art. 450] 4 and 17 are the only integral roots the cubic can have, and it is easily seen that 17 is not a root.

Thus $n=4$ and the numbers must therefore be 63 and 65.

75. We have $$p_1=b_1,\ p_2=a_2b_1+b_2.$$

Also $$q_1=a_1,\ q_2=a_1a_2.$$

Assume that all the numerators and denominators up to the n^{th} satisfy the given laws. Then

$$\frac{p_{n+1}}{q_{n+1}}=\frac{a_np_{n-1}+\left(b_n+\dfrac{b_{n+1}}{a_{n+1}}\right)}{q_n}$$

$$=\frac{p_n+\dfrac{b_{n+1}}{a_{n+1}}}{q_n}=\frac{a_{n+1}p_n+b_{n+1}}{a_{n+1}q_n}.$$

Hence, by induction, the given laws hold true universally.

We have $$q_n=(n+1)\,q_{n-1}=(n+1)\,nq_{n-2}=\ldots=\underline{|n+1}.$$

Also $$p_n=(n+1)\,p_{n-1}+n;$$

$$\therefore\ p_n+1=(n+1)\,(p_{n-1}+1),$$

$$p_{n-1}+1=n\,(p_{n-2}+1),$$

$$\ldots\ldots\ldots\ =\ \ldots\ldots\ldots$$

$$p_2+1=3\,(p_1+1),$$

and $$p_1+1=2.$$

Hence $$p_n+1=\underline{|n+1}.$$

Thus $$\frac{p_n}{q_n}=\frac{\underline{|n+1}-1}{\underline{|n+1}},$$

the limit of which is unity.

76. $1+\frac{x}{1}+\frac{x(x+1)}{1.2}+\frac{x(x+1)(x+2)}{1.2.3}+\ldots$

$$=\frac{1}{1}-\frac{x}{x+1}-\frac{x+1}{x+3}-\frac{2(x+2)}{x+5}-\ldots \text{ [Art. 370, C].}$$

Hence the n^{th} convergent to the continued fraction is equal to

$$1+\frac{x}{1}+\frac{x(x+1)}{1.2}+\frac{x(x+1)(x+2)}{1.2.3}+\ldots+\frac{x(x+1)\ldots(x+n-1)}{1.2.3\ldots n}.$$

Then
$$\frac{x}{1}=\frac{1}{x}\left\{\frac{x(x+1)}{1}-x\right\},$$

$$\frac{x(x+1)}{1.2}=\frac{1}{x}\left\{\frac{x(x+1)(x+2)}{1.2}-\frac{x(x+1)}{1}\right\},$$

$$\ldots\ldots = \ldots\ldots$$

$$\frac{x(x+1)\ldots(x+n-1)}{\lfloor n}=\frac{1}{x}\left\{\frac{x(x+1)\ldots(x+n)}{\lfloor n}-\frac{x(x+1)\ldots(x+n-1)}{\lfloor n-1}\right\}.$$

$$\therefore S_n=1+\frac{1}{x}\left\{\frac{x(x+1)\ldots(x+n)}{\lfloor n}-x\right\}$$

$$=(x+1)(x+2)\ldots(x+n)/\lfloor n.$$

77. $$\Sigma^1{}_n=\frac{p_n}{q_n},\quad \Sigma^1{}_{n-1}=\frac{p_{n-1}}{q_{n-1}}.$$

Also [Art. 357, Ex. 1],

$$\Sigma_1{}^n=\frac{p_n}{p_{n-1}} \text{ and } \Sigma_2{}^n=\frac{q_n}{q_{n-1}}.$$

Hence
$$\Sigma_n{}^1\times\Sigma_2{}^n=\frac{p_n}{q_n}\times\frac{q_n}{q_{n-1}}=\frac{p_n}{q_{n-1}}$$

$$=\frac{p_n}{p_{n-1}}\times\frac{p_{n-1}}{q_{n-1}}=\Sigma^n{}_1\times\Sigma^1{}_{n-1}.$$

78. Since $p_n=p_{n-1}+a_np_{n-2},\ q_n=q_{n-1}+a_nq_{n-2}$,

we have
$$\frac{q_n}{q_{n-1}}=1+\frac{a_n}{q_{n-1}/q_{n-2}}$$

$$=1+\frac{a_n}{1}+\frac{a_{n-1}}{1}+\ldots+\frac{a_2}{q_2/q_1}$$

$$=1+\frac{a_n}{1}+\frac{a_{n-1}}{1}+\ldots+\frac{a_2}{1}+\frac{a_1}{1}$$

$$=1+p'_{n-1}/q'_{n-1}.$$

Hence $\frac{q_n}{q_{n-1}} = \frac{p'_{n-1}+q'_{n-1}}{q'_{n-1}}$, whence it follows that

$$q_n = p'_{n-1} + q'_{n-1},$$

and

$$q_{n-1} = q'_{n-1}.$$

Similarly, $\frac{p_n}{p_{n-1}} = 1 + \frac{a_n}{1} + \frac{a_{n-1}}{1} + \ldots + \frac{a_2}{a_1}$

$$= 1 + \frac{a_1 p'_{n-2}}{a_1 q'_{n-2}} \text{ [since no cancelling is allowed].}$$

Hence

$$p_{n-1} = a_1 q'_{n-2}.$$

79. Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \ldots = \frac{1}{a} - \frac{a^2}{a+b} - \frac{b^2}{b+c} - \ldots$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots = \frac{1}{m} - \frac{m^2}{2m+1} - \frac{(m+1)^2}{2m+3} - \ldots.$$

Hence $m - \frac{m^2}{2m+1} - \frac{(m+1)^2}{2m+3} - \ldots$ to infinity $= \frac{1}{\frac{1}{m} + \frac{1}{m+1} + \ldots} = 0.$

80. The general method will be seen to depend on the possibility of finding a rationalising factor for an expression of the form

$$M = a_0 + a_1 x^{\frac{1}{n}} + a_2 x^{\frac{2}{n}} + \ldots a_{n-1} x^{\frac{n-1}{n}}.$$

The rational result can be written down in the form of a determinant by multiplying the equation by $x^{\frac{1}{n}}, x^{\frac{2}{n}}, \ldots, x^{\frac{n-1}{n}}$ in succession and eliminating $x^{\frac{1}{n}}, \ldots, x^{\frac{n-1}{n}}$ from the equations so formed.

In the particular case we have

$$2^{\frac{1}{3}} = 1 + 2^{\frac{1}{3}} - 1 = 1 + \frac{1}{2^{\frac{2}{3}} + 2^{\frac{1}{3}} + 1},$$

$$2^{\frac{2}{3}} + 2^{\frac{1}{3}} + 1 = 3 + 2^{\frac{2}{3}} + 2^{\frac{1}{3}} - 2 = 3 + \frac{10}{4 \cdot 2^{\frac{1}{3}} + 3 \cdot 2^{\frac{2}{3}} + 2},$$

[by using the identity $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$$= a^3 + b^3 + c^3 - 3abc].$$

$$\frac{4 \cdot 2^{\frac{1}{3}} + 3 \cdot 2^{\frac{2}{3}} + 2}{10} = 1 + \frac{4 \cdot 2^{\frac{1}{3}} + 3 \cdot 2^{\frac{2}{3}} - 8}{10}$$

$$= 1 + \frac{30}{59 \cdot 2^{\frac{1}{3}} + 40 \cdot 2^{\frac{2}{3}} + 40}.$$

$$\frac{59 \cdot 2^{\frac{1}{3}} + 40 \cdot 2^{\frac{2}{3}} + 40}{30} = 5 + \frac{59 \cdot 2^{\frac{1}{3}} + 40 \cdot 2^{\frac{2}{3}} - 110}{30}.$$

Hence $$2^{\frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{1} + \frac{1}{5} + \ldots,$$

the convergents to which are $1, \frac{4}{3}, \frac{5}{4}, \frac{29}{23}$.

81. Let $$M = (a-x)(b-y)(c-z)(ax+by+cz),$$

then $$abcM = (a^2 - ax)(b^2 - by)(c^2 - cz)(ax + by + cz).$$

The sum of the factors on the right is now constant; hence, as the factors are all positive, the product is greatest when they are all equal.

Thus M is greatest when

$$a^2 - ax = b^2 - by = c^2 - cz = ax + by + cz.$$

Hence $$a^2 - ax = \frac{1}{4}(a^2 + b^2 + c^2) = b^2 - by = c^2 - cz.$$

Hence the greatest value of M is $(a^2 + b^2 + c^2)^4 / 256abc$.

82. $$ayz + bzx + cxy < 0,$$

if $$a^2yz - (bz + cy)(by + cz) < 0,$$

i.e. if $$bcy^2 + bcz^2 + (b^2 + c^2 - a^2)yz > 0;$$

$\therefore$ if $$\{2bcy + (b^2 + c^2 - a^2)z\}^2 + \{4b^2c^2 - (b^2 + c^2 - a^2)^2\}z^2 > 0,$$

which is obvious, since

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = \{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}$$
$$= (a+b+c)(b+c-a)(a-b+c)(a+b-c),$$

and is therefore positive.

83. We have to prove that if $m > n$ and $x > 1$,

$$n(1 + x + x^2 + \ldots + x^{m-1}) > m(1 + x + x^2 + \ldots + x^{n-1}),$$

or $$n(x^n + x^{n+1} + \ldots + x^{m-1}) > (m-n)(1 + x + \ldots + x^{n-1}).$$

Now $$n(x^n + x^{n+1} + \ldots + x^{m-1}) > n(m-n)x^n,$$

$$n(m-n)x^n > (m-n)nx^{n-1},$$

and $$(m-n)nx^{n-1} > (m-n)(1 + x + \ldots + x^{n-1}).$$

Similarly if $x < 1$.

84. First, if p and q be positive integers.

Then $$q(a^p - 1) > p(a^q - 1),$$

if $$(a-1)\{q(a^{p-1} + a^{p-2} + \ldots + 1) - p(a^{q-1} + a^{q-2} + \ldots + 1)\} > 0,$$

or, if $$(a-1)\{q(a^{p-1} + a^{p-2} + \ldots + a^q) - (p-q)(a^{q-1} + a^{q-2} + \ldots + 1)\} > 0.$$

The last inequality will be true if $a > 1$, since

$$(a-1)\{q(p-q)a^q - (p-q)qa^{q-1}\} > 0,$$

and it will be true if $a < 1$, since

$$(1-a)\{(p-q)qa^{q-1} - q(p-q)a^q\} > 0.$$

Next if p **and** q (or either of them) be fractional. **Suppose**

$$p=\frac{m}{r} \text{ and } q=\frac{n}{r},$$

where m, n, r are integers and $m>n$.

Then $$q\,(a^p-1)>p\,(a^q-1),$$

if $$n\,(a^{\frac{m}{r}}-1)>m\,(a^{\frac{n}{r}}-1),$$

i.e. if $n\,(b^m-1)>m\,(b^n-1)$, [where $b^r=a$].

But, since m and n are positive integers, the last inequality is true for all positive values of b by the first case.

Next let p and q be both negative. Suppose $p=-m$ and $q=-n$; then

$$m<n, \text{ since } p>q.$$

Then $$q\,(a^p-1)>p\,(a^q-1),$$

if $$-n\,(a^{-m}-1)>-m\,(a^{-n}-1),$$

if $$m\,(a^{-n}-1)>n\,(a^{-m}-1),$$

or, putting $$b=\frac{1}{a},$$

if $$m\,(b^n-1)>n\,(b^m-1),$$

which is true by the preceding cases since m, n, b are positive and $m<n$.

Lastly, suppose $p>0>q$, so that q alone is negative and equal to $-m$, suppose.

Then $$q\,(a^p-1)<p\,(a^q-1),$$

if $$-m\,(a^p-1)<p\,(1-a^m)/a^m,$$

i.e. if $$p\,(a^m-1)<ma^m\,(a^p-1),$$

or if $$(a-1)\,\{p\,(a^{m-1}+a^{m-2}+\ldots+1)-ma^m\,(a^{p-1}+a^{p-2}+\ldots+1)\}<0.$$

Now the last inequality is true when $a>1$, since

$$(a-1)\,\{p\,.\,ma^{m-1}-ma^m\,.\,p\}<0;$$

and the inequality is true when $a<1$, since

$$(a-1)\,\{pma^{m-1}-ma^mp\}<0.$$

85. If $x>1$, $x^x>x$; $\therefore\ x^{-x}<x^{-1}$.

Also if $x<1$, $x^x>x$; $\therefore\ x^{-x}<x^{-1}$.

And, by Ex. 84, since x is positive

$$\frac{x^{x+1}-1}{x+1}>\frac{x^x-1}{x};$$

$$\therefore\ 1>x^x\,(1+x-x^x),$$

i.e. $x^{-x}>1+x-x^2$.

Hence $$x^{-1}>x^{-x}>1+x-x^2.$$

86. If m and n are integers,

$$\frac{\frac{x}{m}+\frac{x}{m}+\ldots \text{ to } m \text{ terms}+\frac{y}{n}+\frac{y}{n}+\ldots \text{ to } n \text{ terms}}{m+n} \leqslant \left\{\left(\frac{x}{m}\right)^m\left(\frac{y}{n}\right)^n\right\}^{\frac{1}{m+n}};$$

$$\therefore \left(\frac{x+y}{m+n}\right)^{m+n} \leqslant \left(\frac{x}{m}\right)^m\left(\frac{y}{n}\right)^n.$$

If m and n be fractional, let pm and pn be integral; then by the above

$$\left(\frac{x+y}{pm+pn}\right)^{pm+pn} \leqslant \left(\frac{x}{pm}\right)^{pm}\left(\frac{y}{pn}\right)^{pn},$$

whence

$$\left(\frac{x+y}{m+n}\right)^{m+n} \leqslant \left(\frac{x}{m}\right)^m\left(\frac{y}{n}\right)^n.$$

The least value of $x+y$ is therefore $(m+n)\{a/(m^m n^n)\}^{\frac{1}{m+n}}$, and $x+y$ has this value when $\frac{x}{m}=\frac{y}{n}$.

87.

$$\left\{\frac{\left(\frac{x}{a}+\frac{x}{a}+\ldots \text{ to } a \text{ terms}\right)+\left(\frac{y}{b}+\frac{y}{b}+\ldots \text{ to } b \text{ terms}\right)+\left(\frac{z}{c}+\frac{z}{c}+\ldots \text{ to } c \text{ terms}\right)}{a+b+c}\right\}^{a+b+c}$$

$$> \left(\frac{x}{a}\right)^a\left(\frac{y}{b}\right)^b\left(\frac{z}{c}\right)^c,$$

except when

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

Hence

$$\frac{(x+y+z)^{a+b+c}}{x^a y^b z^c} > \frac{(a+b+c)^{a+b+c}}{a^a b^b c^c},$$

except when

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

88. Since

$$a>\alpha,\ b>\beta,\ c>\gamma,\ d>\delta,$$

$$(a-\alpha)(b-\beta)>0;$$

$$\therefore ab+\alpha\beta>b\alpha+a\beta;$$

$$\therefore 2(ab+\alpha\beta)>ab+\alpha\beta+b\alpha+a\beta>(a+\alpha)(b+\beta).$$

Similarly

$$2(cd+\gamma\delta)>(c+\gamma)(d+\delta),$$

and

$$2(ab\,.\,cd+\alpha\beta\,.\,\gamma\delta)>(ab+\alpha\beta)(cd+\gamma\delta).$$

Hence

$$8abcd+8\alpha\beta\gamma\delta>(a+\alpha)(b+\beta)(c+\gamma)(d+\delta).$$

89.

$$n\Sigma a_1 b_1 - \Sigma a_1\,.\,\Sigma b_1 = (n-1)\Sigma a_1 b_1 - \Sigma a_1 b_2$$

$$= \Sigma(a_r b_r - a_r b_s - a_s b_r + a_s b_s)$$

$$= \Sigma(a_r - a_s)(b_r - b_s).$$

But $a_r - a_s$ and $b_r - b_s$ are both positive or both negative, and therefore the product is always positive.

Hence $$n\Sigma a_1 b_1 > \Sigma a_1 \cdot \Sigma b_1,$$

or $$\frac{\Sigma a_1 b_1}{n} > \frac{\Sigma a_1}{n} \cdot \frac{\Sigma b_1}{n}.$$

Similarly $$\frac{\Sigma a_1 b_1 \cdot c_1}{n} > \frac{\Sigma a_1 b_1}{n} \cdot \frac{\Sigma c_1}{n};$$

$$\therefore \frac{\Sigma a_1 b_1 c_1}{n} > \frac{\Sigma a_1}{n} \cdot \frac{\Sigma b_1}{n} \cdot \frac{\Sigma c_1}{n}.$$

90. Take a quantities each $\frac{1}{a}$, b quantities each $\frac{1}{b}$, and c quantities each $\frac{1}{c}$.

Then $$\left[\frac{\left(\frac{1}{a}+\frac{1}{a}+\dots\right)+\left(\frac{1}{b}+\frac{1}{b}+\dots\right)+\left(\frac{1}{c}+\frac{1}{c}+\dots\right)}{a+b+c}\right]^{a+b+c}$$

$$> \left(\frac{1}{a}\right)^a \left(\frac{1}{b}\right)^b \left(\frac{1}{c}\right)^c,$$

i.e. $$\left(\frac{3}{a+b+c}\right)^{a+b+c} > \frac{1}{a^a b^b c^c};$$

$$\therefore a^a b^b c^c > \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

Again, taking a quantities each $b+c$, we have

$$\left[\frac{(b+c)\,a+(c+a)\,b+(a+b)\,c}{a+b+c}\right]^{a+b+c} > (b+c)^a\,(c+a)^b\,(a+b)^c.$$

But $$\frac{3\Sigma bc}{\Sigma a} < \Sigma a;$$

$$\therefore \left[\frac{2}{3}\Sigma a\right]^{a+b+c} > \left[\frac{2\Sigma bc}{\Sigma a}\right]^{a+b+c} > (b+c)^a\,(c+a)^b\,(a+b)^c.$$

91. By Fermat's Theorem $a^{19-1}-1 = M\,(19)$,

and $$a^{18}-1 = M\,(a^6-1) = M\,(7).$$

Since a is *odd*, put $a = 2n+1$.

Then $$(1+2n)^{18}-1 = \left\{1+18\,.\,2n+\frac{18\,.\,17}{1\,.\,2}\,2^2n^2+M\,(8)\right\}-1$$

$$= M\,(8).$$

Since a is a prime, we may suppose $a = 3p \pm 1$, and therefore $a^2 = M\,(3)+1$.

Hence $$a^{18}-1 = \{M\,(3)+1\}^9-1 = \{1+M\,(9)\}-1 = M\,(9).$$

Hence, as 19, 7, 8 and 9 are prime to one another,

$$a^{18}-1 = M\,(19\times 7\times 8\times 9) = M\,(9576).$$

92. We have to prove that $N^{mnp+1} - N =$ multiple of $(m+1)(n+1)(p+1)$.

Now, since $m+1$ is a prime number, $N^m = 1 + M(m+1)$, unless N is a multiple of $m+1$.

But, if $N^m = 1 + M(m+1)$, then $N^{mnp} = 1 + M(m+1)$.

Thus $$N(N^{mnp} - 1) = M(m+1).$$

Similarly $$N \times (N^{mnp} - 1) = M(n+1),$$

and $$N(N^{mnp} - 1) = M(p+1).$$

Hence $$N \times (N^{mnp} - 1) = M\{(m+1)(n+1)(p+1)\},$$

since $m+1$, $n+1$, $p+1$ are prime to one another.

93. The possible values of z are 0 to $2n$ inclusive.

We can give to z any integral value from 0 to $2n$ inclusive.

If z has the *even* value $2r$, we have

$$x + 2y = 6(n - r),$$

and this equation can be satisfied by giving to y any one of the values from 0 to $3(n-r)$. Thus, when $z = 2r$, the original equation has $3(n-r)+1$ solutions.

Hence the total number of solutions when z is even is

$$\Sigma_{r=n}^{r=0}\{3(n-r)+1\} = \frac{1}{2}(n+1)(2+3n).$$

Again, if z has the *odd* value $2r+1$, we have

$$x + 2y = 6(n - r) - 3,$$

and this equation can be satisfied by giving to y any one of the values from 0 to $3(n-r)-2$. Thus, when $z = 2r+1$, the original equation has $3(n-r)-1$ solutions. Hence the total number of solutions when n is odd is

$$\Sigma_{r=n-1}^{r=0}\{3(n-r)-1\} = \frac{1}{2}n(3n+1).$$

Hence the whole number of solutions

$$= \frac{1}{2}(n+1)(3n+2) + \frac{1}{2}n(3n+1) = 3n^2 + 3n + 1.$$

94. Since $2 = M(2)$, $3 \cdot 4 = M(2^2)$, $4 \cdot 5 \cdot 6 = M(2^3)$, the theorem is true when n is 1 or 2 or 3.

Assume that the theorem holds good for any particular value of n; then

$$F(n+1) = F(n)(2n+1)(2n+2)/(n+1)$$
$$= 2F(n) \cdot (2n+1).$$

Hence, if $F(n) = M(2^n)$, then $F(n+1) = M(2^{n+1})$,

whence the proposition follows by induction.

95. Let a be any one of the prime factors of $\underline{|n}$ which is prime to q. Then, if $p+rq$ and $p+sq$ leave the same remainder when divided by a, we have $(p+rq)-(p+sq)=M(a)$, that is $(r-s)q=M(a)$. Hence $r-s$ must be a multiple of q. It therefore follows that one out of every a consecutive factors of the numerator is divisible by a, and therefore there will be $I\left(\frac{n}{a}\right)$ factors *at least* of the numerator divisible by a. Also if $p+\kappa q$ be the smallest of these, and, if $p+\kappa q=aa$, the factors of the numerator which are divisible by a will, after division by a, be a, $a+q$, $a+2q$, ..., $a+I\left(\frac{n}{a}\right)q$, where a is prime to q.

The number of these which are again divisible by a can as before be shewn to be $I\left(\frac{n}{a^2}\right)$.

By continuing this reasoning we find that the number of times the numerator contains the factor a is at least

$$I\left(\frac{n}{a}\right)+I\left(\frac{n}{a^2}\right)+I\left(\frac{n}{a^3}\right)+\ldots$$

But we know that the number of times $\underline{|n}$ is divisible by any prime factor a is

$$I\left(\frac{n}{a}\right)+I\left(\frac{n}{a^2}\right)+I\left(\frac{n}{a^3}\right)+\ldots$$

Thus every prime factor contained in $\underline{|n}$, except those which are not prime to q, that is except q itself, since q is a prime, is contained to at least as high a power in $(p+q)(p+2q)\ldots(p+\overline{n-1}\,q)$. This proves the theorem.

96. $$\underline{|p-1}=(p-1)(p-2)\ldots(p-\overline{p-1})$$
$$=p^{p-1}-s_1p^{p-2}+\ldots+s_{p-3}\cdot p^2-s_{p-2}p+\underline{|p-1}.$$

Hence $$p^{p-2}-s_1p^{p-3}+\ldots+S_{p-3}p-S_{p-2}=0.$$

Now all the terms except the two last are divisible by p^2; also by Lagrange's Theorem [Art. 394] S_{p-3} is divisible by p. Whence it follows that s_{p-2} is divisible by p^2.

97. By Art. 439 the sum of the r^{th} powers of any number of quantities can be expressed in terms of lower powers and of the products one together, two together, &c. up to r together.

Now by Lagrange's theorem the sum of the products r together of the numbers $1, 2, \ldots, p-1$ will be divisible by p if p be prime and r be not greater than $p-2$.

It therefore follows that the sum of the r^{th} powers of the numbers 1, 2, 3, ..., $p-1$ will be divisible by p provided r be not greater than $p-2$.

98. $$\underline{|2p-2}\,\underline{|p}\,\underline{|p}=2p\,\underline{|p}\,\{(2p-1)(2p-2)\ldots(p+1)-\underline{|p-1}\}$$
$$=M(p^2)\{(2p-1)(2p-2)\ldots(2p-\overline{p-1})-\underline{|p-1}\}$$
$$=M(p^2)\{(2p)^{p-1}-S_1(2p)^{p-2}+\ldots+S_{p-3}(2p)^2-S_{p-2}(2p)\}.$$

Now all the terms of

$$(2p)^{p-1} - S_1 (2p)^{p-2} + \ldots + S_{p-3} (2p)^2 - S_{p-2} (2p)$$

except the two last are clearly divisible by p^2, and since S_{p-3} is divisible by p and S_{p-2} by p^2 [by 96], it follows that all the terms are divisible by p^3.

99. The chance of drawing the b black balls in the first b drawing and then $a-b$ white balls will be

$$\frac{b}{a+b} \cdot \frac{b-1}{a+b-1} \cdot \frac{b-2}{a-b-2} \cdots \frac{1}{a+1} \cdot \frac{a}{a} \cdot \frac{a-1}{a-1} \cdots \frac{a-b}{a-b}.$$

And it is easily seen that the chance is the same of drawing the black balls in any other order, for the numerator will be *the product of the same factors* although not in the same order, and the denominator will be precisely the same in whatever order the balls may be drawn.

Hence the chance that the b black balls may be drawn in any order whatever is $\underline{|b}\,\underline{|a} / \underline{|a+b}$.

But the number of different orders in which the b black balls can be drawn is

$$\frac{\underline{|a}}{\underline{|b}\,\underline{|a-b}}.$$

Hence the chance required is

$$\frac{\underline{|b}\,\underline{|a}}{\underline{|a+b}} \times \frac{\underline{|a}}{\underline{|b}\,\underline{|a-b}} = \frac{\underline{|a}\,\underline{|a}}{\underline{|a+b}\,\underline{|a-b}}.$$

100. The bag may have contained 8, 7, 6, 5, 4, 3 or 2 sovereigns, and these cases are all assumed to be equally likely.

The chances of drawing in succession two coins which are not sovereigns out of a bag containing 10 coins of which 8, 7, 6, 5, 4, 3, 2 are sovereigns are respectively

$$\frac{2}{90}, \frac{6}{90}, \frac{12}{90}, \frac{20}{90}, \frac{30}{90}, \frac{42}{90} \text{ and } \frac{56}{90}.$$

Hence [Art. 412] the chance that the bag only contained 2 sovereigns is

$$\frac{56}{90} \Big/ \left(\frac{2}{90} + \frac{6}{90} + \frac{12}{90} + \frac{20}{90} + \frac{30}{90} + \frac{42}{90} + \frac{56}{90}\right) = \frac{1}{3}.$$

101. This is included in Ex. 1, Art. 414, r being equal to 1.

102. Since $\alpha+\beta+\gamma=0$, and $\alpha\beta\gamma=G$,

$$(\alpha-\beta)(\alpha-\gamma) = \alpha^2 - \alpha(\beta+\gamma) + \beta\gamma$$

$$= 2\alpha^2 + \beta\gamma = 2\alpha^2 - \frac{G}{\alpha}.$$

Put $y = 2\alpha^2 - \frac{G}{\alpha}$, and eliminate α between this equation and

$$\alpha^3 + 3H\alpha + G = 0.$$

103. If α_1, α_2, &c. be the roots, we have

$$\alpha_1{}^m + a\alpha_1{}^2 + b = 0;$$

$$\therefore\ \alpha_1{}^{m+1} + a\alpha_1{}^3 + b\alpha_1 = 0,$$

and
$$\alpha_1{}^{2m-1} + a\alpha_1{}^{m+1} + b\alpha_1{}^{m-1} = 0.$$

Hence
$$\Sigma\alpha_1{}^{2m-1} + a\Sigma\alpha_1{}^{m+1} + b\Sigma\alpha_1{}^{m-1} = 0,$$

$$\Sigma\alpha_1{}^{m+1} + a\Sigma\alpha_1{}^3 + b\Sigma\alpha_1 = 0,$$

$$\Sigma\alpha_1{}^{m-1} + a\Sigma\alpha_1 + b\Sigma\frac{1}{\alpha_1} = 0.$$

Now, if $m > 5$,
$$\Sigma\alpha_1 = 0,\ \Sigma\frac{1}{\alpha_1} = 0,$$

$$\Sigma\alpha_1{}^2 = (\Sigma\alpha_1)^2 - 2\Sigma\alpha_1\alpha_2 = 0,$$

and
$$\Sigma\alpha_1{}^3 = 3\Sigma\alpha_1\alpha_2\alpha_3 + \Sigma\alpha_1 \cdot (\Sigma\alpha_1{}^2 - \Sigma\alpha_1\alpha_2) = 0.$$

Hence $\Sigma\alpha_1{}^{m-1} = \Sigma\alpha_1{}^{m+1} = 0$, and therefore $\Sigma\alpha_1{}^{2m-1} = 0$.

104. The given equation must be equivalent to

$$(x^2 - 2\alpha x + \alpha^2 + \beta^2)(x^2 - 2\beta x + \alpha^2 + \beta^2) = 0,$$

i. e. $x^4 - 2(\alpha + \beta)x^3 + \ldots + (\alpha^2 + \beta^2)^2 = 0,$

whence it follows that $\alpha = 2$ and $\beta = 1$.

105. In order that the equation

$$x^{2n+2} + (2 - a)x^{2n+1} + (b - a + 1)x^{2n} + (a + 2bn)x + a + (2n - 1)b = 0,$$

may have two equal roots, it must have a root common to the equation,

$$(2n + 2)x^{2n+1} + (2n + 1)(2 - a)x^{2n} + 2n(b - a + 1)x^{2n-1} + (a + 2bn) = 0.$$

This common root is -1, and dividing the left side of the original equation by the factor $(x + 1)^2$, the remaining roots of the equation are given by

$$x^{2n} - ax^{2n-1} + (a + b)x^{2n-2} - (a + 2b)x^{2n-3} + \ldots$$
$$- \{a + (2n - 2)b\}x + a + (2n - 1)b = 0.$$

Hence
$$S_1 = a,\ S_2 = a + b,\ S_3 = a + 2b,\ \&c.$$

106. The equation may be written

$$x^{n-1}(p_0x - p_1) + x^{n-3}(p_2x - p_3) + \ldots + p_n = 0.$$

Hence if $x > H$, every term of the expression on the left is necessarily positive, and therefore no real root of the equation can be greater than H.

Again, the equation may be written

$$x^n + x^{n-2}(-p_1x + p_2) + x^{n-4}(-p_3x + p_4) + \ldots + (-p_{n-1}x + p_n) = 0.$$

Hence, if $x < K$, every term of the expression on the left is necessarily positive, and therefore no real root of the equation can be less than K. Thus all the real roots must be greater than K and less than H, and therefore there can be no real roots if K be greater than H.

107. Subtract the first column from each of the others; then

$$\begin{vmatrix} 1, & 1, & 1, & 1, & \ldots \\ 1, & 1+a, & 1, & 1, & \ldots \\ 1, & 1, & 1+a, & 1, & \ldots \\ 1, & 1, & 1, & 1+a, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, & 0, & \ldots \\ 1, & a, & 0, & 0, & \ldots \\ 1, & 0, & a, & 0, & \ldots \\ 1, & 0, & 0, & a, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix} = a^{n-1}.$$

108. Subtract the first column from each of the others; then add all the rows to the first; then

$$\begin{vmatrix} 1+a, & 1, & 1, & 1, & \dots \\ 1, & 1+a, & 1, & 1, & \dots \\ 1, & 1, & 1+a, & 1, & \dots \\ 1, & 1, & 1, & 1+a, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} 1+a, & -a, & -a, & -a, & \dots \\ 1, & a, & 0, & 0, & \dots \\ 1, & 0, & a, & 0, & \dots \\ 1, & 0, & 0, & a, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} n+a, & 0, & 0, & 0, & \dots \\ 1, & a, & 0, & 0, & \dots \\ 1, & 0, & a, & 0, & \dots \\ 1, & 0, & 0, & a, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = (n+a)\, a^{n-1}.$$

109. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a' & b' & c' & d' \\ aa' & bb' & cc' & dd' \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} \times \begin{vmatrix} c' & d' \\ cc' & dd' \end{vmatrix} + \&c.$$

$$= (b-a)(d-c)\, c'd' - (c-a)(d-b)\, b'd' + (d-a)(c-b)\, b'c' + (c-b)(d-a)\, a'd' - (d-b)(c-a)\, a'c' + (d-c)(b-a)\, a'b'$$

$$= (a-b)(c-d)(a'b'+c'd') - (a-c)(b-d)(a'c'+b'd') + (a-d)(b-c)(a'd'+b'c')$$

$$= (a-b)(c-d)\{(a'-c')(b'-d') + a'd' + b'c'\} - (a-c)(b-d)\{(a'-b')(c'-d') + a'd' + b'c'\} + (a-d)(b-c)\{\qquad\qquad a'd' + b'c'\}$$

$$= (a-b)(c-d)(a'-c')(b'-d') - (a-c)(b-d)(a'-b')(c'-d'),$$

since $(a-b)(c-d) - (a-c)(b-d) + (a-d)(b-c) = 0.$

110. $$(a^3+b^3+c^3-3abc)^2 = \begin{vmatrix} a, & b, & c \\ c, & a, & b \\ b, & c, & a \end{vmatrix} \times \begin{vmatrix} a, & b, & c \\ c, & a, & b \\ b, & c, & a \end{vmatrix}.$$

111. Subtract the first column from the second and divide the members of that column by $a-b$; also subtract the first column from the third and divide by $a-c$. Then we have

$$\begin{vmatrix} x^2-a^2, & a+b, & a+c \\ (x-a)^3, & (x-a)^2+(x-a)(x-b)+(x-b)^2, & (x-a)^2+(x-a)(x-c)+(x-c)^2 \\ (x+a)^3, & -(x+a)^2-(x+a)(x+b)-(x+b)^2, & -(x+a)^2-(x+a)(x+c)-(x+c)^2 \end{vmatrix} = 0.$$

Subtract the second column from the third, and divide by $b-c$; then we have

$$\begin{vmatrix} x^2-a^2, & a+b, & -1 \\ (x-a)^3, & (x-a)^2+(x-a)(x-b)+(x-b)^2, & 3x-a-b-c \\ (x+a)^3, & -(x+a)^2-(x+a)(x+b)-(x+b)^2, & 3x+a+b+c \end{vmatrix} = 0.$$

Add the third row to the second for a new second row, and take this from twice the third for a new third row. Then

$$\begin{vmatrix} x^2-a^2, & a+b, & -1 \\ 2x(x^2+3a^2), & -6x(a+b), & 6x \\ 2a(a^2+3x^2), & -2(3x^2+a^2+ab+b^2), & 2(a+b+c) \end{vmatrix} = 0;$$

$$\therefore x \begin{vmatrix} x^2-a^2, & a+b, & -1 \\ x^2+3a^2, & -3(a+b), & 3 \\ a(a^2+3x^2), & -3x^2-a^2-ab-b^2, & a+b+c \end{vmatrix} = 0.$$

Multiply the first row by 3 and add the second; then we have

$$x^3 \begin{vmatrix} a+b, & 1 \\ 3x^2+a^2+ab+b^2, & a+b+c \end{vmatrix} = 0,$$

$$\text{i.e. } x^3(3x^2-bc-ca-ab)=0.$$

112.

$$\begin{vmatrix} b^2c^2, & bc, & b+c \\ c^2a^2, & ca, & c+a \\ a^2b^2, & ab, & a+b \end{vmatrix}$$

$$= \Sigma b^2c^2a\{c(a+b)-b(c+a)\}$$

$$= a^2b^2c^2\Sigma(c-b)=0.$$

113. Multiply the first column by $p^2+p(a^2+b^2+c^2)$ and subtract from the last; then we have

$$\begin{vmatrix} 1 & 0 & p & 0 \\ 1 & b+c & bc & -pa^2+b^2c^2 \\ 1 & c+a & ca & -pb^2+c^2a^2 \\ 1 & a+b & ab & -pc^2+a^2b^2 \end{vmatrix}$$

$$= \begin{vmatrix} b+c, & bc, & -pa^2+b^2c^2 \\ c+a, & ca, & -pb^2+c^2a^2 \\ a+b, & ab, & -pc^2+a^2b^2 \end{vmatrix} + p \begin{vmatrix} 1, & b+c, & -pa^2+b^2c^2 \\ 1, & c+a, & -pb^2+c^2a^2 \\ 1, & a+b, & -pc^2+a^2b^2 \end{vmatrix}.$$

The first determinant is by Ex. 112 equal to

$$-p \begin{vmatrix} b+c, & bc, & a^2 \\ c+a, & ca, & b^2 \\ a+b, & ab, & c^2 \end{vmatrix} = p(b-c)(c-a)(a-b)(\Sigma a^2+\Sigma bc).$$

The second determinant is equal to

$$p \begin{vmatrix} 1 & b+c & -pa^2+b^2c^2 \\ 0 & 1 & p(a+b)+c^2(a+b) \\ 0 & 1 & p(b+c)+a^2(b+c) \end{vmatrix} (a-b)(b-c)$$

$$= p(b-c)(c-a)(a-b)\{p-\Sigma bc\};$$

$$\therefore \text{ original determinant} = p(b-c)(c-a)(a-b)\{p+\Sigma a^2\}.$$

114. The determinant is equal to $-b^4c^4\{a^4-(a^2+b^2)(a^2+c^2)\}$

$+a^2b^2(b^2+c^2)(c^2+a^2)\{c^2(a^2+b^2)+a^2b^2\}$

$+a^2c^2(a^2+b^2)(b^2+c^2)\{b^2(c^2+a^2)+c^2a^2\}$

$=\Sigma b^2c^2\{+b^4c^4+a^2b^2(b^2+c^2)(c^2+a^2)+a^2c^2(a^2+b^2)(b^2+c^2)\}$

$=\Sigma b^2c^2\{\Sigma a^4b^4+2a^2b^2c^2\Sigma a^2\}$

$=\{\Sigma b^2c^2\}^3$.

115. $(bc+ca+ab)^2=\begin{vmatrix} 0 & b & c & 1 \\ a & 0 & c & 1 \\ a & b & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 0 & b & c & 1 \\ a & 0 & c & 1 \\ a & b & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$

$=$ given result by Art. 430.

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